# 9.2 Series and Convergence

- Understand the definition of a convergent infinite series.
- Use properties of infinite geometric series.
- Use the $n$th-Term Test for Divergence of an infinite series.

## Infinite Series

**INFINITE SERIES**

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

*If throughout the first half of a given time interval a variation continues at a certain intensity, throughout the next quarter of the interval at double the intensity, throughout the following eighth at triple the intensity and so ad infinitum; then the average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).*

This is the same as saying that the sum of the infinite series

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{n}{2^n}+\cdots$$

is 2.

One important application of infinite sequences is in representing "infinite summations." Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \qquad \text{Infinite series}$$

is an **infinite series** (or simply a **series**). The numbers $a_1$, $a_2$, $a_3$, are the **terms** of the series. For some series it is convenient to begin the index at $n = 0$ (or some other integer). As a typesetting convention, it is common to represent an infinite series as simply $\Sigma\, a_n$. In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the following **sequence of partial sums.**

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

If this sequence of partial sums converges, the series is said to converge and has the sum indicated in the following definition.

**DEFINITIONS OF CONVERGENT AND DIVERGENT SERIES**

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to $S$, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit $S$ is called the **sum of the series.**

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series **diverges.**

**STUDY TIP** As you study this chapter, you will see that there are two basic questions involving infinite series. Does a series converge or does it diverge? If a series converges, what is its sum? These questions are not always easy to answer, especially the second one.

**EXPLORATION**

***Finding the Sum of an Infinite Series*** Find the sum of each infinite series. Explain your reasoning.

**a.** $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

**b.** $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots$

**c.** $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

**d.** $\frac{15}{100} + \frac{15}{10{,}000} + \frac{15}{1{,}000{,}000} + \cdots$

**TECHNOLOGY** Figure 9.5 shows the first 15 partial sums of the infinite series in Example 1(a). Notice how the values appear to approach the line $y = 1$.

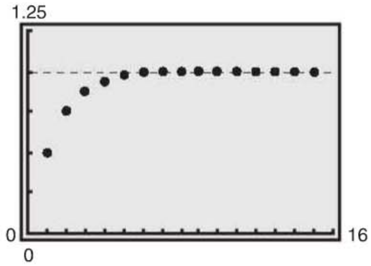



**Figure 9.5**

**NOTE** You can geometrically determine the partial sums of the series in Example 1(a) using Figure 9.6.

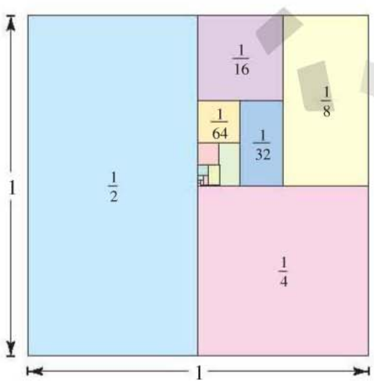



**Figure 9.6**

**FOR FURTHER INFORMATION** To learn more about the partial sums of infinite series, see the article "Six Ways to Sum a Series" by Dan Kalman in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

## EXAMPLE 1 Convergent and Divergent Series

**a.** The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has the following partial sums.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

Because

$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$$

it follows that the series converges and its sum is 1.

**b.** The $n$th partial sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

is given by

$$S_n = 1 - \frac{1}{n+1}.$$

Because the limit of $S_n$ is 1, the series converges and its sum is 1.

**c.** The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$

diverges because $S_n = n$ and the sequence of partial sums diverges. ■

The series in Example 1(b) is a **telescoping series** of the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots.$$ Telescoping series

Note that $b_2$ is canceled by the second term, $b_3$ is canceled by the third term, and so on. Because the $n$th partial sum of this series is

$$S_n = b_1 - b_{n+1}$$

it follows that a telescoping series will converge if and only if $b_n$ approaches a finite number as $n \to \infty$. Moreover, if the series converges, its sum is

$$S = b_1 - \lim_{n\to\infty} b_{n+1}.$$

### EXAMPLE 2 Writing a Series in Telescoping Form

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

**Solution**

Using partial fractions, you can write

$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n - 1)(2n + 1)} = \frac{1}{2n - 1} - \frac{1}{2n + 1}.$$

From this telescoping form, you can see that the $n$th partial sum is

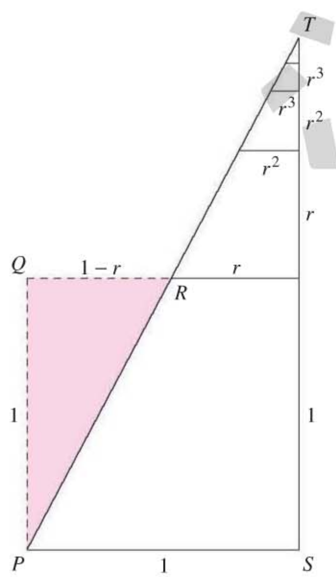
$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdot \cdot \cdot + \left(\frac{1}{2n - 1} - \frac{1}{2n + 1}\right) = 1 - \frac{1}{2n + 1}.$$

So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2n + 1}\right) = 1.$$ ■

**EXPLORATION**

In "Proof Without Words," by Benjamin G. Klein and Irl C. Bivens, the authors present the following diagram. Explain why the final statement below the diagram is valid. How is this result related to Theorem 9.6?




$\Delta PQR \approx \Delta TSP$

$$1 + r + r^2 + r^3 + \cdot \cdot \cdot = \frac{1}{1 - r}$$

Exercise taken from "Proof Without Words" by Benjamin G. Klein and Irl C. Bivens, *Mathematics Magazine*, October 1988, by permission of the authors.

## Geometric Series

The series given in Example 1(a) is a **geometric series.** In general, the series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdot \cdot \cdot + ar^n + \cdot \cdot \cdot, \quad a \neq 0$$ Geometric series



is a **geometric series** with ratio $r$.

**THEOREM 9.6 CONVERGENCE OF A GEOMETRIC SERIES**

A geometric series with ratio $r$ diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad 0 < |r| < 1.$$

**PROOF** It is easy to see that the series diverges if $r = \pm 1$. If $r \neq \pm 1$, then $S_n = a + ar + ar^2 + \cdot \cdot \cdot + ar^{n-1}$. Multiplication by $r$ yields

$$rS_n = ar + ar^2 + ar^3 + \cdot \cdot \cdot + ar^n.$$

Subtracting the second equation from the first produces $S_n - rS_n = a - ar^n$. Therefore, $S_n(1 - r) = a(1 - r^n)$, and the $n$th partial sum is

$$S_n = \frac{a}{1 - r}(1 - r^n).$$

If $0 < |r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$, and you obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\frac{a}{1 - r}(1 - r^n)\right] = \frac{a}{1 - r}\left[\lim_{n\to\infty} (1 - r^n)\right] = \frac{a}{1 - r}$$

which means that the series *converges* and its sum is $a/(1 - r)$. It is left to you to show that the series diverges if $|r| > 1$. ■

**TECHNOLOGY** Try using a graphing utility or writing a computer program to compute the sum of the first 20 terms of the sequence in Example 3(a). You should obtain a sum of about 5.999994.

### EXAMPLE 3 Convergent and Divergent Geometric Series

**a.** The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$$
$$= 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdot \cdot \cdot$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$. Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1-r} = \frac{3}{1-(1/2)} = 6.$$

**b.** The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdot \cdot \cdot$$

has a ratio of $r = \frac{3}{2}$. Because $|r| \geq 1$, the series diverges. ■

The formula for the sum of a geometric series can be used to write a repeating decimal as the ratio of two integers, as demonstrated in the next example.

### EXAMPLE 4 A Geometric Series for a Repeating Decimal



Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

**Solution** For the repeating decimal $0.\overline{08}$, you can write

$$0.080808 \ldots = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \cdot \cdot \cdot$$
$$= \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n.$$

For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So,

$$0.080808 \ldots = \frac{a}{1-r} = \frac{8/10^2}{1-(1/10^2)} = \frac{8}{99}.$$

Try dividing 8 by 99 on a calculator to see that it produces $0.\overline{08}$. ■

The convergence of a series is not affected by removal of a finite number of terms from the beginning of the series. For instance, the geometric series

$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

both converge. Furthermore, because the sum of the second series is $a/(1-r) = 2$, you can conclude that the sum of the first series is

$$S = 2 - \left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right]$$
$$= 2 - \frac{15}{8} = \frac{1}{8}.$$

STUDY TIP As you study this chapter, it is important to distinguish between an infinite series and a sequence. A sequence is an ordered collection of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

whereas a series is an infinite sum of terms from a sequence

$$a_1 + a_2 + \cdots + a_n + \cdots.$$

The following properties are direct consequences of the corresponding properties of limits of sequences.

**THEOREM 9.7 PROPERTIES OF INFINITE SERIES**

Let $\Sigma a_n$ and $\Sigma b_n$ be convergent series, and let $A$, $B$, and $c$ be real numbers. If $\Sigma a_n = A$ and $\Sigma b_n = B$, then the following series converge to the indicated sums.

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = cA$
2. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

## *n*th-Term Test for Divergence

The following theorem states that if a series converges, the limit of its $n$th term must be 0.

NOTE Be sure you see that the converse of Theorem 9.8 is generally not true. That is, if the sequence $\{a_n\}$ converges to 0, then the series $\Sigma a_n$ may either converge or diverge.

**THEOREM 9.8 LIMIT OF THE *n*TH TERM OF A CONVERGENT SERIES**

If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.



PROOF Assume that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

Then, because $S_n = S_{n-1} + a_n$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$$

it follows that

$$\begin{aligned} L = \lim_{n\to\infty} S_n &= \lim_{n\to\infty} (S_{n-1} + a_n) \\ &= \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n \\ &= L + \lim_{n\to\infty} a_n \end{aligned}$$

which implies that $\{a_n\}$ converges to 0. ■

The contrapositive of Theorem 9.8 provides a useful test for *divergence*. This ***n*th-Term Test for Divergence** states that if the limit of the $n$th term of a series does *not* converge to 0, the series must diverge.

**THEOREM 9.9 *n*TH-TERM TEST FOR DIVERGENCE**

If $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

### EXAMPLE 5 Using the *n*th-Term Test for Divergence

**a.** For the series $\sum_{n=0}^{\infty} 2^n$, you have

$$\lim_{n\to\infty} 2^n = \infty.$$

So, the limit of the $n$th term is not 0, and the series diverges.

**b.** For the series $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$, you have

$$\lim_{n\to\infty} \frac{n!}{2n! + 1} = \frac{1}{2}.$$

So, the limit of the $n$th term is not 0, and the series diverges.

**c.** For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Because the limit of the $n$th term is 0, the $n$th-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence. (In the next section, you will see that this particular series diverges.)

**STUDY TIP** The series in Example 5(c) will play an important role in this chapter.

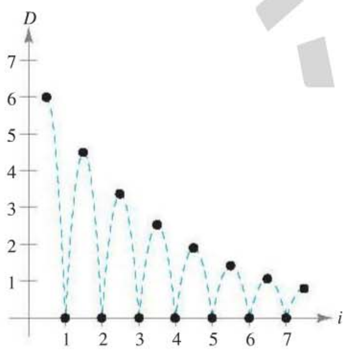
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

You will see that this series diverges even though the $n$th term approaches 0 as $n$ approaches $\infty$.



### EXAMPLE 6 Bouncing Ball Problem

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.




The height of each bounce is three-fourths the height of the preceding bounce.
**Figure 9.7**

**Solution** When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let $D_i$ be the distance traveled up and down. For example, $D_2$ and $D_3$ are as follows.

$$D_2 = \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)$$

$$D_3 = \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)^2$$

By continuing this process, it can be determined that the total vertical distance is

$$\begin{aligned} D &= 6 + 12\left(\tfrac{3}{4}\right) + 12\left(\tfrac{3}{4}\right)^2 + 12\left(\tfrac{3}{4}\right)^3 + \cdots \\ &= 6 + 12\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^{n+1} \\ &= 6 + 12\left(\tfrac{3}{4}\right)\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^{n} \\ &= 6 + 9\left(\frac{1}{1 - \frac{3}{4}}\right) \\ &= 6 + 9(4) \\ &= 42 \text{ feet.} \end{aligned}$$

■

## 9.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**In Exercises 1–6, find the sequence of partial sums $S_1$, $S_2$, $S_3$, $S_4$, and $S_5$.**

**1.** $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

**2.** $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \frac{5}{6 \cdot 7} + \cdots$

**3.** $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} - \cdots$

**4.** $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$

**5.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2^{n-1}}$

**6.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

**In Exercises 7 and 8, determine whether $\{a_n\}$ and $\sum a_n$ are convergent.**

**7.** $a_n = \dfrac{n+1}{n}$

**8.** $a_n = 3\left(\dfrac{4}{5}\right)^n$

**In Exercises 9–18, verify that the infinite series diverges.**

**9.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{7}{6}\right)^n$

**10.** $\displaystyle\sum_{n=0}^{\infty} 5\left(\frac{11}{10}\right)^n$

**11.** $\displaystyle\sum_{n=0}^{\infty} 1000(1.055)^n$

**12.** $\displaystyle\sum_{n=0}^{\infty} 2(-1.03)^n$

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

**14.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+3}$

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

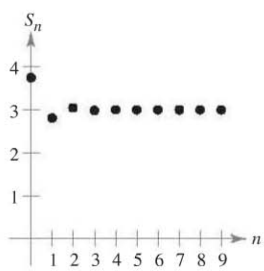
**16.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

**17.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$

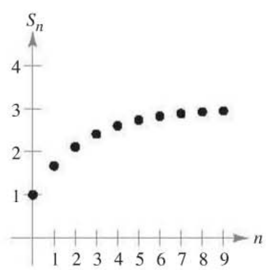
**18.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$

**In Exercises 19–24, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).] Use the graph to estimate the sum of the series. Confirm your answer analytically.**

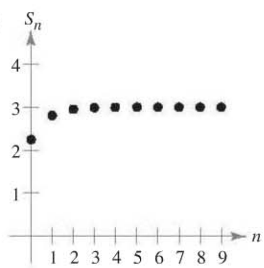
(a)



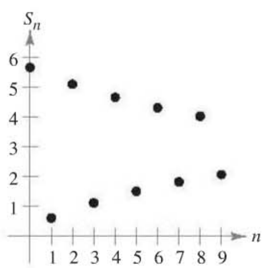
(b)



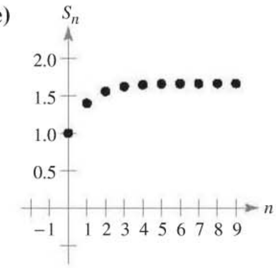
(c)



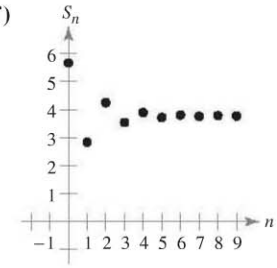
(d)



(e)



(f)



**19.** $\displaystyle\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n$

**20.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

**21.** $\displaystyle\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n$

**22.** $\displaystyle\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{8}{9}\right)^n$

**23.** $\displaystyle\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{1}{2}\right)^n$

**24.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$

**In Exercises 25–30, verify that the infinite series converges.**

**25.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n$

**26.** $\displaystyle\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n$

**27.** $\displaystyle\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$

**28.** $\displaystyle\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots$

**29.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (Use partial fractions.)

**30.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (Use partial fractions.)

***Numerical, Graphical, and Analytic Analysis*** **In Exercises 31–36, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum $S_n$ and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.**

| $n$ | 5 | 10 | 20 | 50 | 100 |
|---|---|---|---|---|---|
| $S_n$ | | | | | |

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$

**32.** $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$

**33.** $\displaystyle\sum_{n=1}^{\infty} 2(0.9)^{n-1}$

**34.** $\displaystyle\sum_{n=1}^{\infty} 3(0.85)^{n-1}$

**35.** $\displaystyle\sum_{n=1}^{\infty} 10(0.25)^{n-1}$

**36.** $\displaystyle\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1}$

**In Exercises 37–52, find the sum of the convergent series.**

**37.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

**38.** $\displaystyle\sum_{n=0}^{\infty} 6\left(\frac{4}{5}\right)^n$

39. $\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n$

40. $\sum_{n=0}^{\infty}3\left(-\frac{6}{7}\right)^n$

41. $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$

42. $\sum_{n=1}^{\infty}\frac{4}{n(n+2)}$

43. $\sum_{n=1}^{\infty}\frac{8}{(n+1)(n+2)}$

44. $\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}$

45. $1 + 0.1 + 0.01 + 0.001 + \cdots$

46. $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$

47. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$

48. $4 - 2 + 1 - \frac{1}{2} + \cdots$

49. $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right)$

50. $\sum_{n=1}^{\infty}[(0.7)^n + (0.9)^n]$

51. $\sum_{n=1}^{\infty}(\sin 1)^n$

52. $\sum_{n=1}^{\infty}\frac{1}{9n^2+3n-2}$

**In Exercises 53–58, (a) write the repeating decimal as a geometric series and (b) write its sum as the ratio of two integers.**

53. $0.\overline{4}$

54. $0.\overline{9}$

55. $0.\overline{81}$

56. $0.\overline{01}$

57. $0.0\overline{75}$

58. $0.2\overline{15}$

**In Exercises 59–76, determine the convergence or divergence of the series.**

59. $\sum_{n=0}^{\infty}(1.075)^n$

60. $\sum_{n=0}^{\infty}\frac{3^n}{1000}$

61. $\sum_{n=1}^{\infty}\frac{n+10}{10n+1}$

62. $\sum_{n=1}^{\infty}\frac{4n+1}{3n-1}$

63. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$

64. $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

65. $\sum_{n=1}^{\infty}\frac{1}{n(n+3)}$

66. $\sum_{n=1}^{\infty}\left(\frac{1}{2n(n+1)}\right)$

67. $\sum_{n=1}^{\infty}\frac{3n-1}{2n+1}$

68. $\sum_{n=1}^{\infty}\frac{3^n}{n^3}$

69. $\sum_{n=0}^{\infty}\frac{4}{2^n}$

70. $\sum_{n=0}^{\infty}\frac{3}{5^n}$

71. $\sum_{n=2}^{\infty}\frac{n}{\ln n}$

72. $\sum_{n=1}^{\infty}\ln\frac{1}{n}$

73. $\sum_{n=1}^{\infty}\left(1+\frac{k}{n}\right)^n$

74. $\sum_{n=1}^{\infty}e^{-n}$

75. $\sum_{n=1}^{\infty}\arctan n$

76. $\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$

## WRITING ABOUT CONCEPTS

77. State the definitions of convergent and divergent series.

78. Describe the difference between $\lim_{n\to\infty} a_n = 5$ and $\sum_{n=1}^{\infty} a_n = 5$.

79. Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.

80. State the $n$th-Term Test for Divergence.

## WRITING ABOUT CONCEPTS (continued)

81. Explain any differences among the following series.

(a) $\sum_{n=1}^{\infty} a_n$ (b) $\sum_{k=1}^{\infty} a_k$ (c) $\sum_{n=1}^{\infty} a_k$

82. (a) You delete a finite number of terms from a divergent series. Will the new series still diverge? Explain your reasoning.

(b) You add a finite number of terms to a convergent series. Will the new series still converge? Explain your reasoning.

**In Exercises 83–90, find all values of $x$ for which the series converges. For these values of $x$, write the sum of the series as a function of $x$.**

83. $\sum_{n=1}^{\infty}\frac{x^n}{2^n}$

84. $\sum_{n=1}^{\infty}(3x)^n$

85. $\sum_{n=1}^{\infty}(x-1)^n$

86. $\sum_{n=0}^{\infty}4\left(\frac{x-3}{4}\right)^n$

87. $\sum_{n=0}^{\infty}(-1)^n x^n$

88. $\sum_{n=0}^{\infty}(-1)^n x^{2n}$

89. $\sum_{n=0}^{\infty}\left(\frac{1}{x}\right)^n$

90. $\sum_{n=1}^{\infty}\left(\frac{x^2}{x^2+4}\right)^n$

**In Exercises 91 and 92, find the value of $c$ for which the series equals the indicated sum.**

91. $\sum_{n=2}^{\infty}(1+c)^{-n} = 2$

92. $\sum_{n=0}^{\infty}e^{cn} = 5$

93. ***Think About It*** Consider the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

Given $x = -1$ and $x = 2$, can you conclude that either of the following statements is true? Explain your reasoning.

(a) $\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$

(b) $-1 = 1 + 2 + 4 + 8 + \cdots$

## CAPSTONE

94. ***Think About It*** Are the following statements true? Why or why not?

(a) Because $\frac{1}{n^4}$ approaches 0 as $n$ approaches $\infty$,

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = 0.$$

(b) Because $\lim_{n\to\infty}\frac{1}{\sqrt[4]{n}} = 0$, the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n}}$ converges.

**In Exercises 95 and 96, (a) find the common ratio of the geometric series, (b) write the function that gives the sum of the series, and (c) use a graphing utility to graph the function and the partial sums $S_3$ and $S_5$. What do you notice?**

95. $1 + x + x^2 + x^3 + \cdots$

96. $1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots$

**In Exercises 97 and 98, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum of the series.**

| | *Function* | *Series* |
|---|---|---|
| **97.** | $f(x) = 3\left[\dfrac{1-(0.5)^x}{1-0.5}\right]$ | $\displaystyle\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$ |
| **98.** | $f(x) = 2\left[\dfrac{1-(0.8)^x}{1-0.8}\right]$ | $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$ |

***Writing*** **In Exercises 99 and 100, use a graphing utility to determine the first term that is less than 0.0001 in each of the convergent series. Note that the answers are very different. Explain how this will affect the rate at which the series converges.**

**99.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$ **100.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}, \quad \sum_{n=1}^{\infty} (0.01)^n$

**101.** ***Marketing*** An electronic games manufacturer producing a new product estimates the annual sales to be 8000 units. Each year 5% of the units that have been sold will become inoperative. So, 8000 units will be in use after 1 year, $[8000 + 0.95(8000)]$ units will be in use after 2 years, and so on. How many units will be in use after $n$ years?

**102.** ***Depreciation*** A company buys a machine for \$475,000 that depreciates at a rate of 30% per year. Find a formula for the value of the machine after $n$ years. What is its value after 5 years?

**103.** ***Multiplier Effect*** The total annual spending by tourists in a resort city is \$200 million. Approximately 75% of that revenue is again spent in the resort city, and of that amount approximately 75% is again spent in the same city, and so on. Write the geometric series that gives the total amount of spending generated by the \$200 million and find the sum of the series.

**104.** ***Multiplier Effect*** Repeat Exercise 103 if the percent of the revenue that is spent again in the city decreases to 60%.

**105.** ***Distance*** A ball is dropped from a height of 16 feet. Each time it drops $h$ feet, it rebounds $0.81h$ feet. Find the total distance traveled by the ball.

**106.** ***Time*** The ball in Exercise 105 takes the following times for each fall.

$s_1 = -16t^2 + 16, \qquad s_1 = 0 \text{ if } t = 1$

$s_2 = -16t^2 + 16(0.81), \qquad s_2 = 0 \text{ if } t = 0.9$

$s_3 = -16t^2 + 16(0.81)^2, \qquad s_3 = 0 \text{ if } t = (0.9)^2$

$s_4 = -16t^2 + 16(0.81)^3, \qquad s_4 = 0 \text{ if } t = (0.9)^3$

$\vdots \qquad \vdots$

$s_n = -16t^2 + 16(0.81)^{n-1}, \qquad s_n = 0 \text{ if } t = (0.9)^{n-1}$

Beginning with $s_2$, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is given by $t = 1 + 2\sum_{n=1}^{\infty}(0.9)^n$. Find this total time.

***Probability*** **In Exercises 107 and 108, the random variable $n$ represents the number of units of a product sold per day in a store. The probability distribution of $n$ is given by $P(n)$. Find the probability that two units are sold in a given day $[P(2)]$ and show that $P(0) + P(1) + P(2) + P(3) + \cdot \cdot \cdot = 1$.**

**107.** $P(n) = \dfrac{1}{2}\left(\dfrac{1}{2}\right)^n$ **108.** $P(n) = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$

**109.** ***Probability*** A fair coin is tossed repeatedly. The probability that the first head occurs on the $n$th toss is given by $P(n) = \left(\frac{1}{2}\right)^n$, where $n \geq 1$.

(a) Show that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 1$.

(b) The expected number of tosses required until the first head occurs in the experiment is given by $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$. Is this series geometric?

CAS (c) Use a computer algebra system to find the sum in part (b).

**110.** ***Probability*** In an experiment, three people toss a fair coin one at a time until one of them tosses a head. Determine, for each person, the probability that he or she tosses the first head. Verify that the sum of the three probabilities is 1.

**111.** ***Area*** The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded (see figure). Determine the area of the shaded regions (a) if this process is continued five more times and (b) if this pattern of shading is continued infinitely.

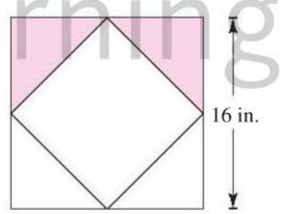



**Figure for 111**

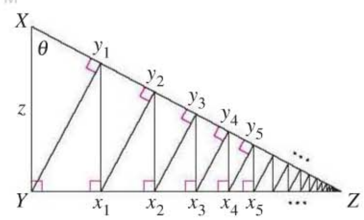



**Figure for 112**

**112.** ***Length*** A right triangle $XYZ$ is shown above where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle, as shown in the figure.

(a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdot \cdot \cdot$ in terms of $z$ and $\theta$.

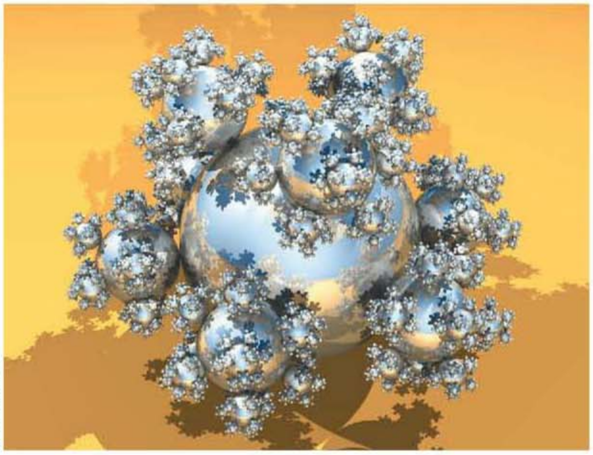
(b) If $z = 1$ and $\theta = \pi/6$, find the total length of the perpendicular line segments.

**In Exercises 113–116, use the formula for the $n$th partial sum of a geometric series**

$$\sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r}.$$

**113.** ***Present Value*** The winner of a \$2,000,000 sweepstakes will be paid \$100,000 per year for 20 years. The money earns 6% interest per year. The present value of the winnings is $\displaystyle\sum_{n=1}^{20} 100{,}000\left(\frac{1}{1.06}\right)^n$. Compute the present value and interpret its meaning.

**114.** ***Sphereflake*** The sphereflake shown below is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely. Prove that the sphereflake has an infinite surface area.



Eric Haines

**115.** ***Salary*** You go to work at a company that pays \$0.01 for the first day, \$0.02 for the second day, \$0.04 for the third day, and so on. If the daily wage keeps doubling, what would your total income be for working (a) 29 days, (b) 30 days, and (c) 31 days?

**116.** ***Annuities*** When an employee receives a paycheck at the end of each month, $P$ dollars is invested in a retirement account. These deposits are made each month for $t$ years and the account earns interest at the annual percentage rate $r$. If the interest is compounded monthly, the amount $A$ in the account at the end of $t$ years is

$$A = P + P\left(1 + \frac{r}{12}\right) + \cdots + P\left(1 + \frac{r}{12}\right)^{12t-1}$$
$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right].$$

If the interest is compounded continuously, the amount $A$ in the account after $t$ years is

$$A = P + Pe^{r/12} + Pe^{2r/12} + Pe^{(12t-1)r/12}$$
$$= \frac{P(e^{rt} - 1)}{e^{r/12} - 1}.$$

Verify the formulas for the sums given above.

***Annuities*** **In Exercises 117–120, consider making monthly deposits of $P$ dollars in a savings account at an annual interest rate $r$. Use the results of Exercise 116 to find the balance $A$ after $t$ years if the interest is compounded (a) monthly and (b) continuously.**

**117.** $P = \$45$, $r = 3\%$, $t = 20$ years

**118.** $P = \$75$, $r = 5.5\%$, $t = 25$ years

**119.** $P = \$100$, $r = 4\%$, $t = 35$ years

**120.** $P = \$30$, $r = 6\%$, $t = 50$ years

**121.** ***Salary*** You accept a job that pays a salary of \$50,000 for the first year. During the next 39 years you receive a 4% raise each year. What would be your total compensation over the 40-year period?

**122.** ***Salary*** Repeat Exercise 121 if the raise you receive each year is 4.5%. Compare the results.

***True or False?*** **In Exercises 123–128, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**123.** If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

**124.** If $\sum_{n=1}^{\infty} a_n = L$, then $\sum_{n=0}^{\infty} a_n = L + a_0$.

**125.** If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^n = \frac{a}{(1-r)}$.

**126.** The series $\sum_{n=1}^{\infty} \frac{n}{1000(n+1)}$ diverges.

**127.** $0.75 = 0.749999 \ldots$

**128.** Every decimal with a repeating pattern of digits is a rational number.

**129.** Show that the series $\sum_{n=1}^{\infty} a_n$ can be written in the telescoping form

$$\sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)]$$

where $S_0 = 0$ and $S_n$ is the $n$th partial sum.

**130.** Let $\Sigma\, a_n$ be a convergent series, and let

$$R_N = a_{N+1} + a_{N+2} + \cdots$$

be the remainder of the series after the first $N$ terms. Prove that $\lim_{N\to\infty} R_N = 0$.

**131.** Find two divergent series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma(a_n + b_n)$ converges.

**132.** Given two infinite series $\Sigma\, a_n$ and $\Sigma\, b_n$ such that $\Sigma\, a_n$ converges and $\Sigma\, b_n$ diverges, prove that $\Sigma(a_n + b_n)$ diverges.

**133.** Suppose that $\Sigma\, a_n$ diverges and $c$ is a nonzero constant. Prove that $\Sigma\, ca_n$ diverges.

**134.** If $\sum_{n=1}^{\infty} a_n$ converges where $a_n$ is nonzero, show that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

**135.** The Fibonacci sequence is defined recursively by $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ and $a_2 = 1$.

(a) Show that $\frac{1}{a_{n+1}a_{n+3}} = \frac{1}{a_{n+1}a_{n+2}} - \frac{1}{a_{n+2}a_{n+3}}$.

(b) Show that $\sum_{n=0}^{\infty} \frac{1}{a_{n+1}a_{n+3}} = 1$.

**136.** Find the values of $x$ for which the infinite series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + \cdots$$

converges. What is the sum when the series converges?

**137.** Prove that $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots = \frac{1}{r-1}$, for $|r| > 1$.