8.4 Trigonometric Substitution

- **Use trigonometric substitution to solve an integral.**
- **Use integrals to model and solve real-life applications.**

Trigonometric Substitution

Now that you can evaluate integrals involving powers of trigonometric functions, you can use **trigonometric substitution** to evaluate integrals involving the radicals

$$\sqrt{a^2 - u^2}, \qquad \sqrt{a^2 + u^2}, \qquad \text{and} \qquad \sqrt{u^2 - a^2}.$$

The objective with trigonometric substitution is to eliminate the radical in the integrand. You do this by using the Pythagorean identities

$$\cos^2\theta = 1 - \sin^2\theta, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \text{and} \quad \tan^2\theta = \sec^2\theta - 1.$$

For example, if $a > 0$, let $u = a\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then

$$\begin{aligned}\sqrt{a^2 - u^2} &= \sqrt{a^2 - a^2\sin^2\theta}\\ &= \sqrt{a^2(1 - \sin^2\theta)}\\ &= \sqrt{a^2\cos^2\theta}\\ &= a\cos\theta.\end{aligned}$$

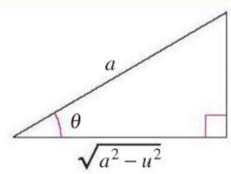
Note that $\cos\theta \ge 0$, because $-\pi/2 \le \theta \le \pi/2$.

TRIGONOMETRIC SUBSTITUTION ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a\sin\theta.$$

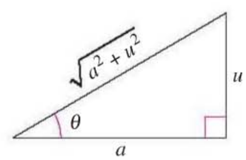
Then $\sqrt{a^2 - u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let

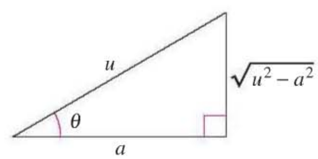
$$u = a\tan\theta.$$

Then $\sqrt{a^2 + u^2} = a\sec\theta$, where $-\pi/2 < \theta < \pi/2$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let

$$u = a\sec\theta.$$



Then

$$\sqrt{u^2 - a^2} = \begin{cases} a\tan\theta \text{ if } u > a, \text{ where } 0 \le \theta < \pi/2\\ -a\tan\theta \text{ if } u < -a, \text{ where } \pi/2 < \theta \le \pi\end{cases}$$

NOTE The restrictions on θ ensure that the function that defines the substitution is one-to-one. In fact, these are the same intervals over which the arcsine, arctangent, and arcsecant are defined.

EXPLORATION

Integrating a Radical Function Up to this point in the text, you have not evaluated the following integral.

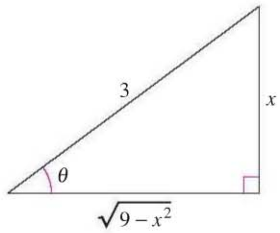
$$\int_{-1}^{1} \sqrt{1 - x^2}\,dx$$

From geometry, you should be able to find the exact value of this integral—what is it? Using numerical integration, with Simpson's Rule or the Trapezoidal Rule, you can't be sure of the accuracy of the approximation. Why?

Try finding the exact value using the substitution

$$x = \sin\theta \text{ and } dx = \cos\theta\,d\theta.$$

Does your answer agree with the value you obtained using geometry?



$\sin\theta = \dfrac{x}{3}$, $\cot\theta = \dfrac{\sqrt{9-x^2}}{x}$

Figure 8.6

EXAMPLE 1 Trigonometric Substitution: $u = a\sin\theta$

Find $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Solution First, note that none of the basic integration rules applies. To use trigonometric substitution, you should observe that $\sqrt{9-x^2}$ is of the form $\sqrt{a^2-u^2}$. So, you can use the substitution

$$x = a\sin\theta = 3\sin\theta.$$

Using differentiation and the triangle shown in Figure 8.6, you obtain

$$dx = 3\cos\theta\,d\theta,\qquad \sqrt{9-x^2} = 3\cos\theta,\qquad \text{and}\qquad x^2 = 9\sin^2\theta.$$

So, trigonometric substitution yields

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{9-x^2}} &= \int \frac{3\cos\theta\,d\theta}{(9\sin^2\theta)(3\cos\theta)} && \text{Substitute.}\\
&= \frac{1}{9}\int \frac{d\theta}{\sin^2\theta} && \text{Simplify.}\\
&= \frac{1}{9}\int \csc^2\theta\,d\theta && \text{Trigonometric identity}\\
&= -\frac{1}{9}\cot\theta + C && \text{Apply Cosecant Rule.}\\
&= -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C && \text{Substitute for } \cot\theta.\\
&= -\frac{\sqrt{9-x^2}}{9x} + C.
\end{aligned}$$

Note that the triangle in Figure 8.6 can be used to convert the θ's back to x's, as follows.

$$\begin{aligned}
\cot\theta &= \frac{\text{adj.}}{\text{opp.}}\\
&= \frac{\sqrt{9-x^2}}{x}
\end{aligned}$$

TECHNOLOGY Use a computer algebra system to find each indefinite integral.

$$\int \frac{dx}{\sqrt{9-x^2}}\qquad \int \frac{dx}{x\sqrt{9-x^2}}\qquad \int \frac{dx}{x^2\sqrt{9-x^2}}\qquad \int \frac{dx}{x^3\sqrt{9-x^2}}$$

Then use trigonometric substitution to duplicate the results obtained with the computer algebra system.

In an earlier chapter, you saw how the inverse hyperbolic functions can be used to evaluate the integrals

$$\int \frac{du}{\sqrt{u^2 \pm a^2}},\qquad \int \frac{du}{a^2-u^2},\qquad \text{and}\qquad \int \frac{du}{u\sqrt{a^2 \pm u^2}}.$$

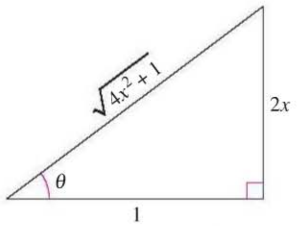
You can also evaluate these integrals using trigonometric substitution. This is shown in the next example.

EXAMPLE 2 Trigonometric Substitution: $u = a \tan \theta$

Find $\displaystyle\int \frac{dx}{\sqrt{4x^2 + 1}}$.

Solution Let $u = 2x$, $a = 1$, and $2x = \tan \theta$, as shown in Figure 8.7. Then,

$$dx = \frac{1}{2} \sec^2 \theta \, d\theta \qquad \text{and} \qquad \sqrt{4x^2 + 1} = \sec \theta.$$



$\tan \theta = 2x$, $\sec \theta = \sqrt{4x^2 + 1}$
Figure 8.7

Trigonometric substitution produces

$$\int \frac{1}{\sqrt{4x^2 + 1}} \, dx = \frac{1}{2} \int \frac{\sec^2 \theta \, d\theta}{\sec \theta} \qquad \text{Substitute.}$$

$$= \frac{1}{2} \int \sec \theta \, d\theta \qquad \text{Simplify.}$$

$$= \frac{1}{2} \ln|\sec \theta + \tan \theta| + C \qquad \text{Apply Secant Rule.}$$

$$= \frac{1}{2} \ln\left|\sqrt{4x^2 + 1} + 2x\right| + C. \qquad \text{Back-substitute.}$$

Try checking this result with a computer algebra system. Is the result given in this form or in the form of an inverse hyperbolic function? ■

You can extend the use of trigonometric substitution to cover integrals involving expressions such as $(a^2 - u^2)^{n/2}$ by writing the expression as

$$(a^2 - u^2)^{n/2} = \left(\sqrt{a^2 - u^2}\right)^n.$$

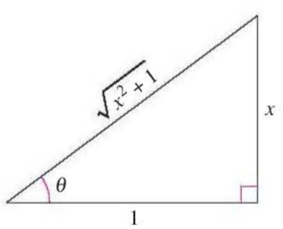
EXAMPLE 3 Trigonometric Substitution: Rational Powers

Find $\displaystyle\int \frac{dx}{(x^2 + 1)^{3/2}}$.

Solution Begin by writing $(x^2 + 1)^{3/2}$ as $\left(\sqrt{x^2 + 1}\right)^3$. Then, let $a = 1$ and $u = x = \tan \theta$, as shown in Figure 8.8. Using

$$dx = \sec^2 \theta \, d\theta \qquad \text{and} \qquad \sqrt{x^2 + 1} = \sec \theta$$

you can apply trigonometric substitution, as follows.



$\tan \theta = x$, $\sin \theta = \dfrac{x}{\sqrt{x^2 + 1}}$
Figure 8.8

$$\int \frac{dx}{(x^2 + 1)^{3/2}} = \int \frac{dx}{\left(\sqrt{x^2 + 1}\right)^3} \qquad \text{Rewrite denominator.}$$

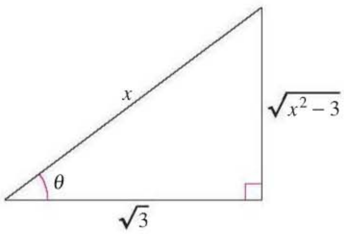
$$= \int \frac{\sec^2 \theta \, d\theta}{\sec^3 \theta} \qquad \text{Substitute.}$$

$$= \int \frac{d\theta}{\sec \theta} \qquad \text{Simplify.}$$

$$= \int \cos \theta \, d\theta \qquad \text{Trigonometric identity}$$

$$= \sin \theta + C \qquad \text{Apply Cosine Rule.}$$

$$= \frac{x}{\sqrt{x^2 + 1}} + C \qquad \text{Back-substitute.}$$ ■



$\sec\theta = \dfrac{x}{\sqrt{3}},\ \tan\theta = \dfrac{\sqrt{x^2-3}}{\sqrt{3}}$

Figure 8.9

For definite integrals, it is often convenient to determine the integration limits for θ that avoid converting back to x. You might want to review this procedure in Section 4.5, Examples 8 and 9.

EXAMPLE 4 Converting the Limits of Integration

Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$.

Solution Because $\sqrt{x^2-3}$ has the form $\sqrt{u^2-a^2}$, you can consider

$$u = x, \quad a = \sqrt{3}, \quad \text{and} \quad x = \sqrt{3}\sec\theta$$

as shown in Figure 8.9. Then,

$$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta \qquad \text{and} \qquad \sqrt{x^2-3} = \sqrt{3}\tan\theta.$$

To determine the upper and lower limits of integration, use the substitution $x = \sqrt{3}\sec\theta$, as follows.

Lower Limit	*Upper Limit*
When $x = \sqrt{3}$, $\sec\theta = 1$ and $\theta = 0$.	When $x = 2$, $\sec\theta = \dfrac{2}{\sqrt{3}}$ and $\theta = \dfrac{\pi}{6}$.

So, you have

Integration limits for x — Integration limits for θ

$$\begin{aligned}
\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_{0}^{\pi/6} \frac{(\sqrt{3}\tan\theta)(\sqrt{3}\sec\theta\tan\theta)\,d\theta}{\sqrt{3}\sec\theta}\\
&= \int_0^{\pi/6} \sqrt{3}\tan^2\theta\,d\theta\\
&= \sqrt{3}\int_0^{\pi/6} (\sec^2\theta - 1)\,d\theta\\
&= \sqrt{3}\Big[\tan\theta - \theta\Big]_0^{\pi/6}\\
&= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)\\
&= 1 - \frac{\sqrt{3}\pi}{6}\\
&\approx 0.0931.
\end{aligned}$$

■

In Example 4, try converting back to the variable x and evaluating the antiderivative at the original limits of integration. You should obtain

$$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx = \sqrt{3}\,\frac{\sqrt{x^2-3}}{\sqrt{3}} - \operatorname{arcsec}\frac{x}{\sqrt{3}}\Bigg]_{\sqrt{3}}^{2}.$$

When using trigonometric substitution to evaluate definite integrals, you must be careful to check that the values of θ lie in the intervals discussed at the beginning of this section. For instance, if in Example 4 you had been asked to evaluate the definite integral

$$\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2-3}}{x}\,dx$$

then using $u = x$ and $a = \sqrt{3}$ in the interval $\left[-2, -\sqrt{3}\right]$ would imply that $u < -a$. So, when determining the upper and lower limits of integration, you would have to choose θ such that $\pi/2 < \theta \le \pi$. In this case the integral would be evaluated as follows.

$$\begin{aligned}
\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2-3}}{x}\,dx &= \int_{5\pi/6}^{\pi} \frac{\left(-\sqrt{3}\tan\theta\right)\left(\sqrt{3}\sec\theta\tan\theta\right)d\theta}{\sqrt{3}\sec\theta}\\
&= \int_{5\pi/6}^{\pi} -\sqrt{3}\tan^2\theta\,d\theta\\
&= -\sqrt{3}\int_{5\pi/6}^{\pi} (\sec^2\theta - 1)\,d\theta\\
&= -\sqrt{3}\Big[\tan\theta - \theta\Big]_{5\pi/6}^{\pi}\\
&= -\sqrt{3}\left[(0-\pi) - \left(-\frac{1}{\sqrt{3}} - \frac{5\pi}{6}\right)\right]\\
&= -1 + \frac{\sqrt{3}\pi}{6}\\
&\approx -0.0931
\end{aligned}$$

Trigonometric substitution can be used with completing the square. For instance, try evaluating the following integral.

$$\int \sqrt{x^2-2x}\,dx$$

To begin, you could complete the square and write the integral as

$$\int \sqrt{(x-1)^2-1^2}\,dx.$$

Trigonometric substitution can be used to evaluate the three integrals listed in the following theorem. These integrals will be encountered several times in the remainder of the text. When this happens, we will simply refer to this theorem. (In Exercise 85, you are asked to verify the formulas given in the theorem.)

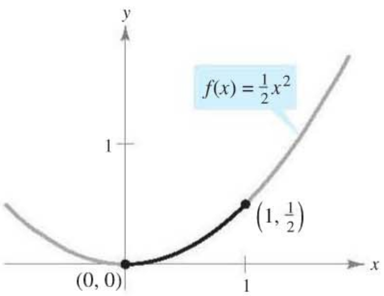
THEOREM 8.2 SPECIAL INTEGRATION FORMULAS ($a > 0$)

1. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(a^2 \arcsin\frac{u}{a} + u\sqrt{a^2-u^2}\right) + C$
2. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2-a^2} - a^2\ln\left|u+\sqrt{u^2-a^2}\right|\right) + C, \quad u > a$
3. $\displaystyle\int \sqrt{u^2+a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2+a^2} + a^2\ln\left|u+\sqrt{u^2+a^2}\right|\right) + C$

Applications

EXAMPLE 5 Finding Arc Length

Find the arc length of the graph of $f(x) = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$ (see Figure 8.10).



The arc length of the curve from $(0, 0)$ to $\left(1, \frac{1}{2}\right)$
Figure 8.10

Solution Refer to the arc length formula in Section 7.4.

$$s = \int_0^1 \sqrt{1 + [f'(x)]^2}\, dx \qquad \text{Formula for arc length}$$

$$= \int_0^1 \sqrt{1 + x^2}\, dx \qquad f'(x) = x$$

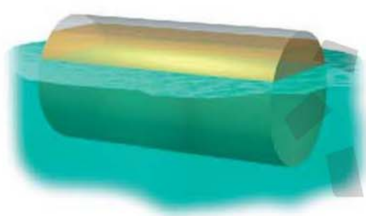
$$= \int_0^{\pi/4} \sec^3 \theta \, d\theta \qquad \text{Let } a = 1 \text{ and } x = \tan\theta.$$

$$= \frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \qquad \text{Example 5, Section 8.2}$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right] \approx 1.148$$

EXAMPLE 6 Comparing Two Fluid Forces

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot), as shown in Figures 8.11 and 8.12. (The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 foot of the barrel is empty.) Compare the fluid forces against one end of the barrel from the inside and from the outside.



The barrel is not quite full of oil—the top 0.2 foot of the barrel is empty.
Figure 8.11

Solution In Figure 8.12, locate the coordinate system with the origin at the center of the circle given by $x^2 + y^2 = 1$. To find the fluid force against an end of the barrel *from the inside*, integrate between -1 and 0.8 (using a weight of $w = 48$).

$$F = w\int_c^d h(y)L(y)\, dy \qquad \text{General equation (see Section 7.7)}$$

$$F_{\text{inside}} = 48\int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2}\, dy$$

$$= 76.8\int_{-1}^{0.8} \sqrt{1 - y^2}\, dy - 96\int_{-1}^{0.8} y\sqrt{1 - y^2}\, dy$$

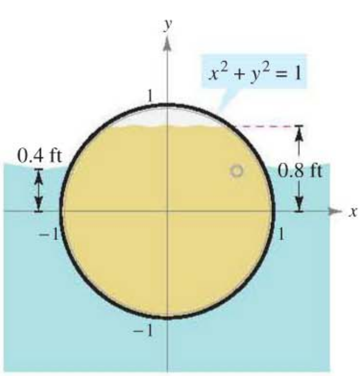


Figure 8.12

To find the fluid force *from the outside*, integrate between -1 and 0.4 (using a weight of $w = 64$).

$$F_{\text{outside}} = 64\int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2}\, dy$$

$$= 51.2\int_{-1}^{0.4} \sqrt{1 - y^2}\, dy - 128\int_{-1}^{0.4} y\sqrt{1 - y^2}\, dy$$

The details of integration are left for you to complete in Exercise 84. Intuitively, would you say that the force from the oil (the inside) or the force from the seawater (the outside) is greater? By evaluating these two integrals, you can determine that

$$F_{\text{inside}} \approx 121.3 \text{ pounds} \qquad \text{and} \qquad F_{\text{outside}} \approx 93.0 \text{ pounds.}$$

8.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, state the trigonometric substitution you would use to find the integral. Do not integrate.

1. $\int (9 + x^2)^{-2}\,dx$
2. $\int \sqrt{4 - x^2}\,dx$
3. $\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$
4. $\int x^2(x^2 - 25)^{3/2}\,dx$

In Exercises 5–8, find the indefinite integral using the substitution $x = 4 \sin \theta$.

5. $\int \frac{1}{(16 - x^2)^{3/2}}\,dx$
6. $\int \frac{4}{x^2\sqrt{16 - x^2}}\,dx$
7. $\int \frac{\sqrt{16 - x^2}}{x}\,dx$
8. $\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$

In Exercises 9–12, find the indefinite integral using the substitution $x = 5 \sec \theta$.

9. $\int \frac{1}{\sqrt{x^2 - 25}}\,dx$
10. $\int \frac{\sqrt{x^2 - 25}}{x}\,dx$
11. $\int x^3\sqrt{x^2 - 25}\,dx$
12. $\int \frac{x^3}{\sqrt{x^2 - 25}}\,dx$

In Exercises 13–16, find the indefinite integral using the substitution $x = \tan \theta$.

13. $\int x\sqrt{1 + x^2}\,dx$
14. $\int \frac{9x^3}{\sqrt{1 + x^2}}\,dx$
15. $\int \frac{1}{(1 + x^2)^2}\,dx$
16. $\int \frac{x^2}{(1 + x^2)^2}\,dx$

In Exercises 17–20, use the Special Integration Formulas (Theorem 8.2) to find the integral.

17. $\int \sqrt{9 + 16x^2}\,dx$
18. $\int \sqrt{4 + x^2}\,dx$
19. $\int \sqrt{25 - 4x^2}\,dx$
20. $\int \sqrt{5x^2 - 1}\,dx$

In Exercises 21–42, find the integral.

21. $\int \frac{x}{\sqrt{x^2 + 36}}\,dx$
22. $\int \frac{x}{\sqrt{36 - x^2}}\,dx$
23. $\int \frac{1}{\sqrt{16 - x^2}}\,dx$
24. $\int \frac{1}{\sqrt{49 - x^2}}\,dx$
25. $\int \sqrt{16 - 4x^2}\,dx$
26. $\int x\sqrt{16 - 4x^2}\,dx$
27. $\int \frac{1}{\sqrt{x^2 - 4}}\,dx$
28. $\int \frac{t}{(4 - t^2)^{3/2}}\,dt$
29. $\int \frac{\sqrt{1 - x^2}}{x^4}\,dx$
30. $\int \frac{\sqrt{4x^2 + 9}}{x^4}\,dx$
31. $\int \frac{1}{x\sqrt{4x^2 + 9}}\,dx$
32. $\int \frac{1}{x\sqrt{4x^2 + 16}}\,dx$
33. $\int \frac{-3x}{(x^2 + 3)^{3/2}}\,dx$
34. $\int \frac{1}{(x^2 + 5)^{3/2}}\,dx$
35. $\int e^{2x}\sqrt{1 + e^{2x}}\,dx$
36. $\int (x + 1)\sqrt{x^2 + 2x + 2}\,dx$
37. $\int e^x\sqrt{1 - e^{2x}}\,dx$
38. $\int \frac{\sqrt{1 - x}}{\sqrt{x}}\,dx$
39. $\int \frac{1}{4 + 4x^2 + x^4}\,dx$
40. $\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\,dx$
41. $\int \operatorname{arcsec} 2x\,dx, \quad x > \frac{1}{2}$
42. $\int x \arcsin x\,dx$

In Exercises 43–46, complete the square and find the integral.

43. $\int \frac{1}{\sqrt{4x - x^2}}\,dx$
44. $\int \frac{x^2}{\sqrt{2x - x^2}}\,dx$
45. $\int \frac{x}{\sqrt{x^2 + 6x + 12}}\,dx$
46. $\int \frac{x}{\sqrt{x^2 - 6x + 5}}\,dx$

In Exercises 47–52, evaluate the integral using (a) the given integration limits and (b) the limits obtained by trigonometric substitution.

47. $\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}}\,dt$
48. $\int_0^{\sqrt{3}/2} \frac{1}{(1 - t^2)^{5/2}}\,dt$
49. $\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}}\,dx$
50. $\int_0^{3/5} \sqrt{9 - 25x^2}\,dx$
51. $\int_4^6 \frac{x^2}{\sqrt{x^2 - 9}}\,dx$
52. $\int_3^6 \frac{\sqrt{x^2 - 9}}{x^2}\,dx$

In Exercises 53 and 54, find the particular solution of the differential equation.

53. $x\frac{dy}{dx} = \sqrt{x^2 - 9}, \quad x \geq 3, \quad y(3) = 1$
54. $\sqrt{x^2 + 4}\,\frac{dy}{dx} = 1, \quad x \geq -2, \quad y(0) = 4$

CAS **In Exercises 55–58, use a computer algebra system to find the integral. Verify the result by differentiation.**

55. $\int \frac{x^2}{\sqrt{x^2 + 10x + 9}}\,dx$
56. $\int (x^2 + 2x + 11)^{3/2}\,dx$
57. $\int \frac{x^2}{\sqrt{x^2 - 1}}\,dx$
58. $\int x^2\sqrt{x^2 - 4}\,dx$

WRITING ABOUT CONCEPTS

59. State the substitution you would make if you used trigonometric substitution and the integral involving the given radical, where $a > 0$. Explain your reasoning.

(a) $\sqrt{a^2 - u^2}$ (b) $\sqrt{a^2 + u^2}$ (c) $\sqrt{u^2 - a^2}$

WRITING ABOUT CONCEPTS (continued)

In Exercises 60 and 61, state the method of integration you would use to perform each integration. Explain why you chose that method. Do not integrate.

60. $\int x\sqrt{x^2+1}\,dx$ **61.** $\int x^2\sqrt{x^2-1}\,dx$

CAPSTONE

62. (a) Evaluate the integral $\int \frac{x}{x^2+9}\,dx$ using u-substitution. Then evaluate using trigonometric substitution. Discuss the results.

(b) Evaluate the integral $\int \frac{x^2}{x^2+9}\,dx$ algebraically using $x^2 = (x^2+9) - 9$. Then evaluate using trigonometric substitution. Discuss the results.

(c) Evaluate the integral $\int \frac{4}{4-x^2}\,dx$ using trigonometric substitution. Then evaluate using the identity $\frac{4}{4-x^2} = \left(\frac{1}{x+2} - \frac{1}{x-2}\right)$. Discuss the results.

True or False? **In Exercises 63–66, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

63. If $x = \sin\theta$, then $\int \frac{dx}{\sqrt{1-x^2}} = \int d\theta$.

64. If $x = \sec\theta$, then $\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \sec\theta\tan\theta\,d\theta$.

65. If $x = \tan\theta$, then $\int_0^{\sqrt{3}} \frac{dx}{(1+x^2)^{3/2}} = \int_0^{4\pi/3} \cos\theta\,d\theta$.

66. If $x = \sin\theta$, then $\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = 2\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$.

67. ***Area*** Find the area enclosed by the ellipse shown in the figure.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

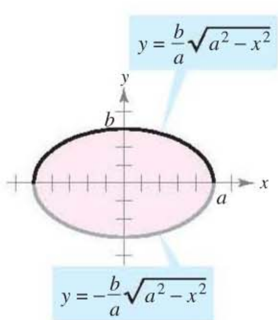


Figure for 67

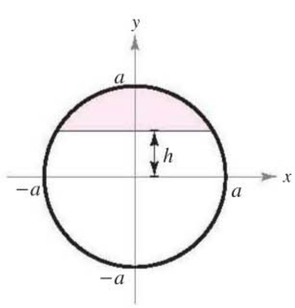


Figure for 68

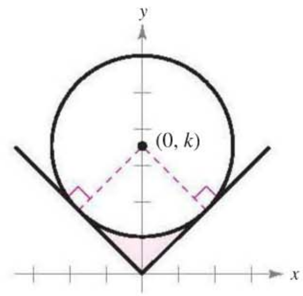
68. ***Area*** Find the area of the shaded region of the circle of radius a, if the chord is h units $(0 < h < a)$ from the center of the circle (see figure).

69. ***Mechanical Design*** The surface of a machine part is the region between the graphs of $y = |x|$ and $x^2 + (y-k)^2 = 25$ (see figure).

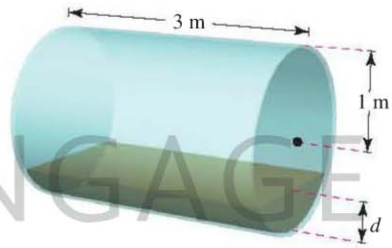
(a) Find k if the circle is tangent to the graph of $y = |x|$.

(b) Find the area of the surface of the machine part.

(c) Find the area of the surface of the machine part as a function of the radius r of the circle.



70. ***Volume*** The axis of a storage tank in the form of a right circular cylinder is horizontal (see figure). The radius and length of the tank are 1 meter and 3 meters, respectively.



(a) Determine the volume of fluid in the tank as a function of its depth d.

(b) Use a graphing utility to graph the function in part (a).

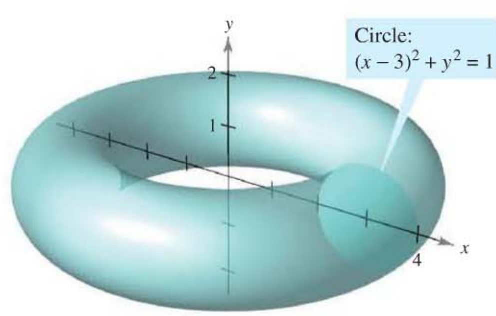
(c) Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

(d) Fluid is entering the tank at a rate of $\frac{1}{4}$ cubic meter per minute. Determine the rate of change of the depth of the fluid as a function of its depth d.

(e) Use a graphing utility to graph the function in part (d). When will the rate of change of the depth be minimum? Does this agree with your intuition? Explain.

Volume of a Torus **In Exercises 71 and 72, find the volume of the torus generated by revolving the region bounded by the graph of the circle about the y-axis.**

71. $(x-3)^2 + y^2 = 1$ (see figure)



72. $(x-h)^2 + y^2 = r^2,\ h > r$

Arc Length **In Exercises 73 and 74, find the arc length of the curve over the given interval.**

73. $y = \ln x, \quad [1, 5]$

74. $y = \frac{1}{2}x^2, \quad [0, 4]$

75. ***Arc Length*** Show that the length of one arch of the sine curve is equal to the length of one arch of the cosine curve.

76. ***Conjecture***

(a) Find formulas for the distances between $(0, 0)$ and (a, a^2) along the line between these points and along the parabola $y = x^2$.

(b) Use the formulas from part (a) to find the distances for $a = 1$ and $a = 10$.

(c) Make a conjecture about the difference between the two distances as a increases.

Projectile Motion **In Exercises 77 and 78, (a) use a graphing utility to graph the path of a projectile that follows the path given by the graph of the equation, (b) determine the range of the projectile, and (c) use the integration capabilities of a graphing utility to determine the distance the projectile travels.**

77. $y = x - 0.005x^2$

78. $y = x - \dfrac{x^2}{72}$

Centroid **In Exercises 79 and 80, find the centroid of the region determined by the graphs of the inequalities.**

79. $y \le 3/\sqrt{x^2 + 9}, \ y \ge 0, \ x \ge -4, \ x \le 4$

80. $y \le \frac{1}{4}x^2, \ (x - 4)^2 + y^2 \le 16, \ y \ge 0$

81. ***Surface Area*** Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, $x = 0$, and $x = \sqrt{2}$ about the x-axis.

82. ***Field Strength*** The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H = \frac{2mL}{(r^2 + L^2)^{3/2}}$$

where $\pm m$ are the poles of the magnet (see figure). Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R}\int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}}\,dr.$$

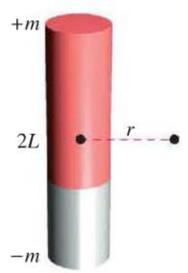


Figure for 82

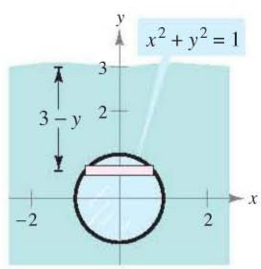


Figure for 83

83. ***Fluid Force*** Find the fluid force on a circular observation window of radius 1 foot in a vertical wall of a large water-filled tank at a fish hatchery when the center of the window is (a) 3 feet and (b) d feet $(d > 1)$ below the water's surface (see figure). Use trigonometric substitution to evaluate the one integral. (Recall that in Section 7.7 in a similar problem, you evaluated one integral by a geometric formula and the other by observing that the integrand was odd.)

84. ***Fluid Force*** Evaluate the following two integrals, which yield the fluid forces given in Example 6.

(a) $F_{\text{inside}} = 48\displaystyle\int_{-1}^{0.8} (0.8 - y)(2)\sqrt{1 - y^2}\,dy$

(b) $F_{\text{outside}} = 64\displaystyle\int_{-1}^{0.4} (0.4 - y)(2)\sqrt{1 - y^2}\,dy$

85. Use trigonometric substitution to verify the integration formulas given in Theorem 8.2.

86. ***Arc Length*** Show that the arc length of the graph of $y = \sin x$ on the interval $[0, 2\pi]$ is equal to the circumference of the ellipse $x^2 + 2y^2 = 2$ (see figure).

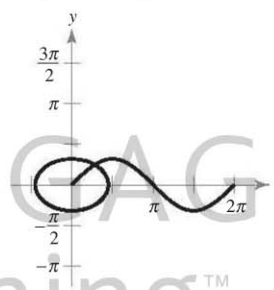


Figure for 86

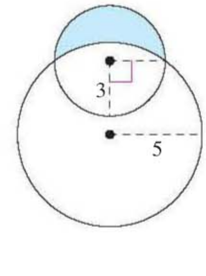
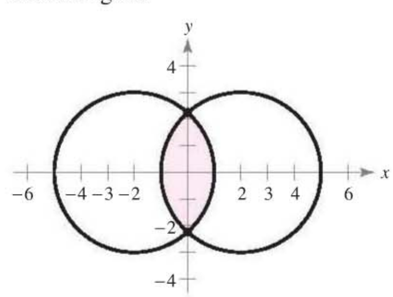


Figure for 87

87. ***Area of a Lune*** The crescent-shaped region bounded by two circles forms a *lune* (see figure). Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.

88. ***Area*** Two circles of radius 3, with centers at $(-2, 0)$ and $(2, 0)$, intersect as shown in the figure. Find the area of the shaded region.



PUTNAM EXAM CHALLENGE

89. Evaluate

$$\int_0^1 \frac{\ln(x + 1)}{x^2 + 1}\,dx.$$