7.1   Area of a Region Between Two Curves

■ Find the area of a region between two curves using integration.
■ Find the area of a region between intersecting curves using integration.
■ Describe integration as an accumulation process.

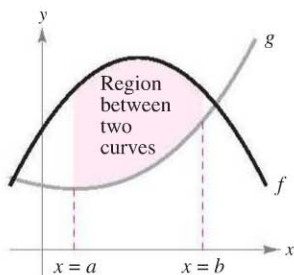## Area of a Region Between Two Curves

With a few modifications, you can extend the application of definite integrals from the area of a region *under* a curve to the area of a region *between* two curves. Consider two functions f and g that are continuous on the interval $[a, b]$. If, as in Figure 7.1, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f, you can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 7.2.
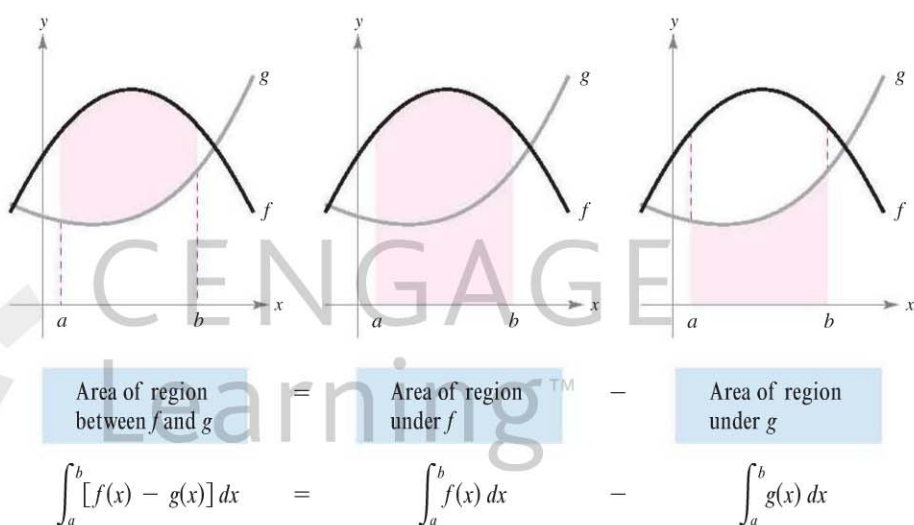

Figure 7.1

Area of region between f and g	=	Area of region under f	−	Area of region under g
$\displaystyle\int_a^b [f(x) - g(x)]\,dx$	=	$\displaystyle\int_a^b f(x)\,dx$	−	$\displaystyle\int_a^b g(x)\,dx$

Figure 7.2

Representative rectangle
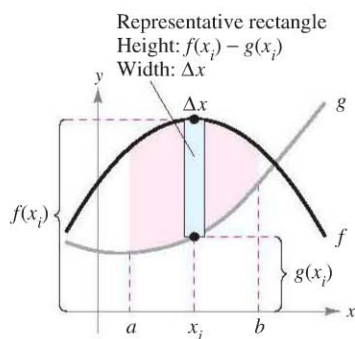Height: $f(x_i) - g(x_i)$
Width: Δx

Figure 7.3

To verify the reasonableness of the result shown in Figure 7.2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx. Then, as shown in Figure 7.3, sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith subinterval. The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\Delta x.$$

By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$, you obtain

$$\lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. So, the area of the given region is

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x$$

$$= \int_a^b [f(x) - g(x)]\,dx.$$

AREA OF A REGION BETWEEN TWO CURVES

If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

In Figure 7.1, the graphs of f and g are shown above the x-axis. This, however, is not necessary. The same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \le f(x)$ for all x in the interval $[a, b]$. This is summarized graphically in Figure 7.4. Notice in Figure 7.4 that the height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the x-axis.
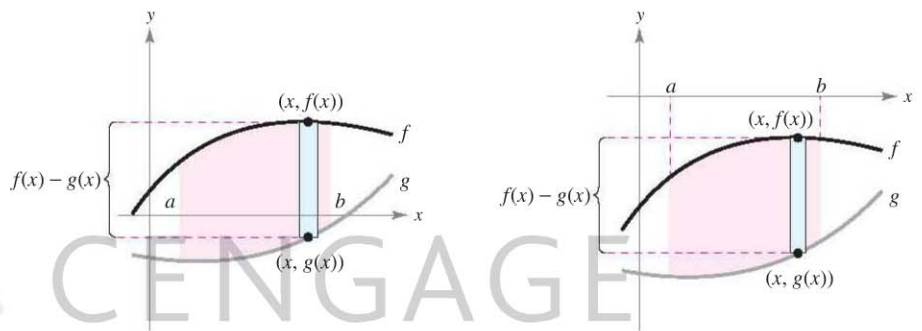

Figure 7.4

Representative rectangles are used throughout this chapter in various applications of integration. A vertical rectangle (of width Δx) implies integration with respect to x, whereas a horizontal rectangle (of width Δy) implies integration with respect to y.

EXAMPLE 1 Finding the Area of a Region Between Two Curves
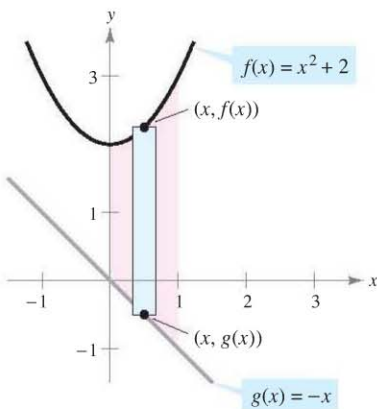
Find the area of the region bounded by the graphs of $f(x) = x^2 + 2$, $g(x) = -x$, $x = 0$, and $x = 1$.

Solution Let $g(x) = -x$ and $f(x) = x^2 + 2$. Then $g(x) \le f(x)$ for all x in $[0, 1]$, as shown in Figure 7.5. So, the area of the representative rectangle is

$$\Delta A = [f(x) - g(x)]\, \Delta x$$
$$= [(x^2 + 2) - (-x)]\, \Delta x$$

and the area of the region is

$$A = \int_a^b [f(x) - g(x)]\, dx = \int_0^1 [(x^2 + 2) - (-x)]\, dx$$
$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^1$$
$$= \frac{1}{3} + \frac{1}{2} + 2$$
$$= \frac{17}{6}.$$

Region bounded by the graph of f, the graph of g, $x = 0$, and $x = 1$.
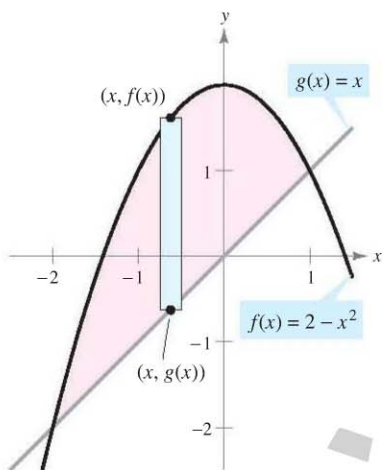**Figure 7.5**

Area of a Region Between Intersecting Curves

In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of a and b are given explicitly. A more common problem involves the area of a region bounded by two *intersecting* graphs, where the values of a and b must be calculated.

EXAMPLE 2   A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

Solution   In Figure 7.6, notice that the graphs of f and g have two points of intersection. To find the x-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for x.

$$2 - x^2 = x \qquad \text{Set } f(x) \text{ equal to } g(x).$$
$$-x^2 - x + 2 = 0 \qquad \text{Write in general form.}$$
$$-(x + 2)(x - 1) = 0 \qquad \text{Factor.}$$
$$x = -2 \text{ or } 1 \qquad \text{Solve for } x.$$

So, $a = -2$ and $b = 1$. Because $g(x) \leq f(x)$ for all x in the interval $[-2, 1]$, the representative rectangle has an area of

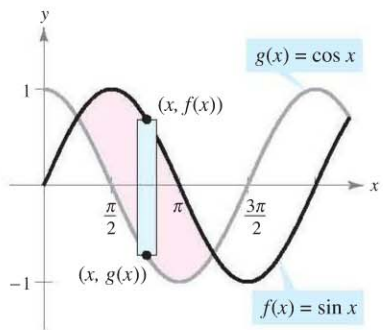$$\Delta A = [f(x) - g(x)]\, \Delta x$$
$$= [(2 - x^2) - x]\, \Delta x$$

and the area of the region is

$$A = \int_{-2}^{1} [(2 - x^2) - x]\, dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1}$$
$$= \frac{9}{2}.$$

$(x, f(x))$

$g(x) = x$

$f(x) = 2 - x^2$

$(x, g(x))$

Region bounded by the graph of f and the graph of g
Figure 7.6

EXAMPLE 3   A Region Lying Between Two Intersecting Graphs

The sine and cosine curves intersect infinitely many times, bounding regions of equal areas, as shown in Figure 7.7. Find the area of one of these regions.

Solution

$$\sin x = \cos x \qquad \text{Set } f(x) \text{ equal to } g(x).$$
$$\frac{\sin x}{\cos x} = 1 \qquad \text{Divide each side by } \cos x.$$
$$\tan x = 1 \qquad \text{Trigonometric identity}$$
$$x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}, \quad 0 \leq x \leq 2\pi \qquad \text{Solve for } x.$$

So, $a = \pi/4$ and $b = 5\pi/4$. Because $\sin x \geq \cos x$ for all x in the interval $[\pi/4, 5\pi/4]$, the area of the region is

$$A = \int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\, dx = \left[-\cos x - \sin x \right]_{\pi/4}^{5\pi/4}$$
$$= 2\sqrt{2}.$$

$g(x) = \cos x$

$(x, f(x))$

$f(x) = \sin x$

$(x, g(x))$

One of the regions bounded by the graphs of the sine and cosine functions
Figure 7.7

If two curves intersect at more than two points, then to find the area of the region between the curves, you must find all points of intersection and check to see which curve is above the other in each interval determined by these points.

## EXAMPLE 4 Curves That Intersect at More Than Two Points

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

Solution   Begin by setting $f(x)$ and $g(x)$ equal to each other and solving for x. This yields the x-values at all points of intersection of the two graphs.

$$3x^3 - x^2 - 10x = -x^2 + 2x \qquad \text{Set } f(x) \text{ equal to } g(x).$$
$$3x^3 - 12x = 0 \qquad \text{Write in general form.}$$
$$3x(x - 2)(x + 2) = 0 \qquad \text{Factor.}$$
$$x = -2, 0, 2 \qquad \text{Solve for } x.$$

So, the two graphs intersect when $x = -2, 0$, and 2. In Figure 7.8, notice that $g(x) \le f(x)$ on the interval $[-2, 0]$. However, the two graphs switch at the origin, and $f(x) \le g(x)$ on the interval $[0, 2]$. So, you need two integrals—one for the interval $[-2, 0]$ and one for the interval $[0, 2]$.

$$A = \int_{-2}^{0} [f(x) - g(x)] \, dx + \int_{0}^{2} [g(x) - f(x)] \, dx$$
$$= \int_{-2}^{0} (3x^3 - 12x) \, dx + \int_{0}^{2} (-3x^3 + 12x) \, dx$$
$$= \left[\frac{3x^4}{4} - 6x^2 \right]_{-2}^{0} + \left[\frac{-3x^4}{4} + 6x^2 \right]_{0}^{2}$$
$$= -(12 - 24) + (-12 + 24) = 24$$
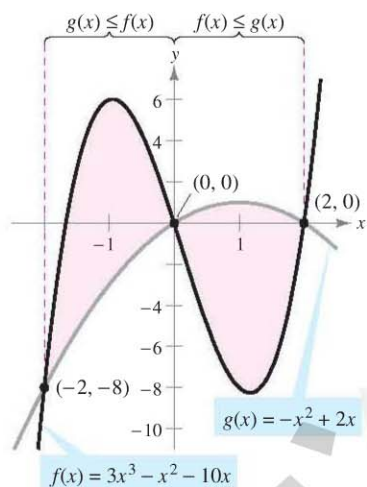
On $[-2, 0]$, $g(x) \le f(x)$, and on $[0, 2]$, $f(x) \le g(x)$
Figure 7.8

NOTE   In Example 4, notice that you obtain an incorrect result if you integrate from -2 to 2. Such integration produces

$$\int_{-2}^{2} [f(x) - g(x)] \, dx = \int_{-2}^{2} (3x^3 - 12x) \, dx = 0.$$

If the graph of a function of y is a boundary of a region, it is often convenient to use representative rectangles that are *horizontal* and find the area by integrating with respect to y. In general, to determine the area between two curves, you can use

$$A = \int_{x_1}^{x_2} [(\text{top curve}) - (\text{bottom curve})] \, dx \qquad \text{Vertical rectangles}$$
$$\underbrace{\phantom{[(\text{top curve}) - (\text{bottom curve})]}}_{\text{in variable } x}$$

$$A = \int_{y_1}^{y_2} [(\text{right curve}) - (\text{left curve})] \, dy \qquad \text{Horizontal rectangles}$$
$$\underbrace{\phantom{[(\text{right curve}) - (\text{left curve})]}}_{\text{in variable } y}$$

where (x_1, y_1) and (x_2, y_2) are either adjacent points of intersection of the two curves involved or points on the specified boundary lines.

The icon ↻ *indicates that you will find a CAS Investigation on the book's website and the online Eduspace® system. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems* Maple *and* Mathematica.

EXAMPLE 5 Horizontal Representative Rectangles

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.
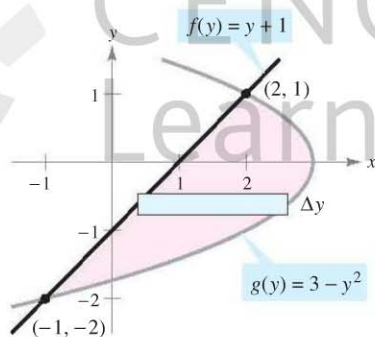
Solution Consider

$$g(y) = 3 - y^2 \quad \text{and} \quad f(y) = y + 1.$$

These two curves intersect when $y = -2$ and $y = 1$, as shown in Figure 7.9. Because $f(y) \le g(y)$ on this interval, you have

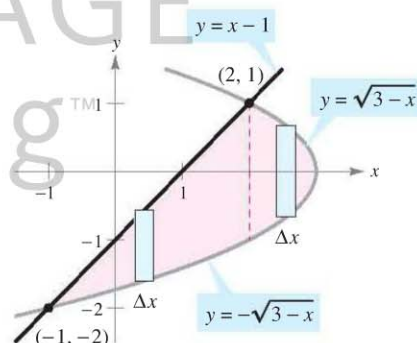$$\Delta A = [g(y) - f(y)] \, \Delta y = [(3 - y^2) - (y + 1)] \, \Delta y.$$

So, the area is

$$
\begin{aligned}
A &= \int_{-2}^{1} [(3 - y^2) - (y + 1)] \, dy \\
&= \int_{-2}^{1} (-y^2 - y + 2) \, dy \\
&= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y \right]_{-2}^{1} \\
&= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) \\
&= \frac{9}{2}.
\end{aligned}
$$

Horizontal rectangles (integration with respect to y)
Figure 7.9

Vertical rectangles (integration with respect to x)
Figure 7.10

In Example 5, notice that by integrating with respect to y you need only one integral. If you had integrated with respect to x, you would have needed two integrals because the upper boundary would have changed at $x = 2$, as shown in Figure 7.10.

$$
\begin{aligned}
A &= \int_{-1}^{2} \left[(x - 1) + \sqrt{3 - x} \right] dx + \int_{2}^{3} \left(\sqrt{3 - x} + \sqrt{3 - x} \right) dx \\
&= \int_{-1}^{2} \left[x - 1 + (3 - x)^{1/2} \right] dx + 2 \int_{2}^{3} (3 - x)^{1/2} \, dx \\
&= \left[\frac{x^2}{2} - x - \frac{(3 - x)^{3/2}}{3/2} \right]_{-1}^{2} - 2\left[\frac{(3 - x)^{3/2}}{3/2} \right]_{2}^{3} \\
&= \left(2 - 2 - \frac{2}{3} \right) - \left(\frac{1}{2} + 1 - \frac{16}{3} \right) - 2(0) + 2\left(\frac{2}{3} \right) \\
&= \frac{9}{2}
\end{aligned}
$$

Integration as an Accumulation Process

In this section, the integration formula for the area between two curves was developed by using a rectangle as the *representative element*. For each new application in the remaining sections of this chapter, an appropriate representative element will be constructed using precalculus formulas you already know. Each integration formula will then be obtained by summing or accumulating these representative elements.

Known precalculus formula	⇒	Representative element	⇒	New integration formula
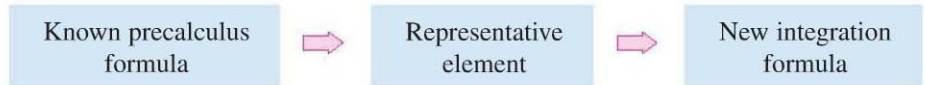
For example, the area formula in this section was developed as follows.

$A = (\text{height})(\text{width})$	⇒	$\Delta A = [f(x) - g(x)] \, \Delta x$	⇒	$A = \int_a^b [f(x) - g(x)] \, dx$
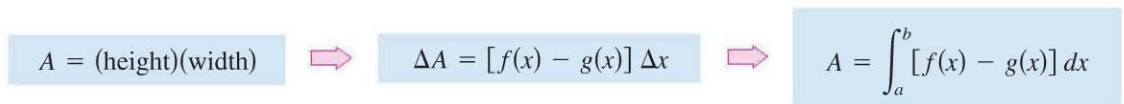
EXAMPLE 6 Describing Integration as an Accumulation Process

Find the area of the region bounded by the graph of $y = 4 - x^2$ and the x-axis. Describe the integration as an accumulation process.

Solution The area of the region is given by

$$A = \int_{-2}^{2} (4 - x^2) \, dx.$$

You can think of the integration as an accumulation of the areas of the rectangles formed as the representative rectangle slides from $x = -2$ to $x = 2$, as shown in Figure 7.11.
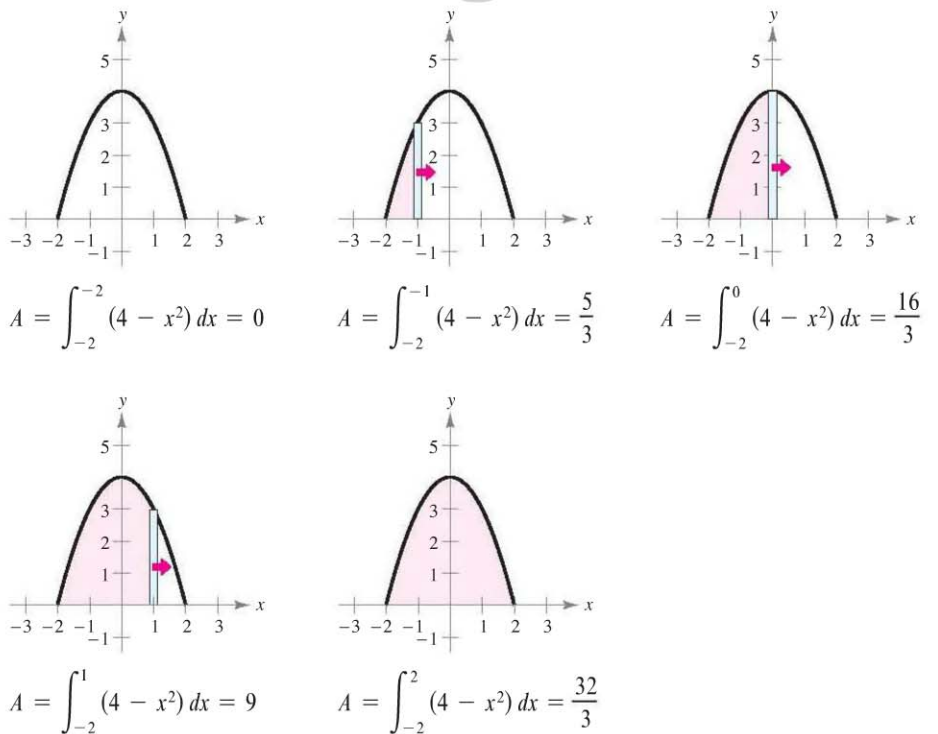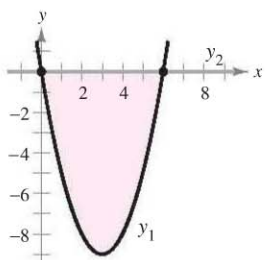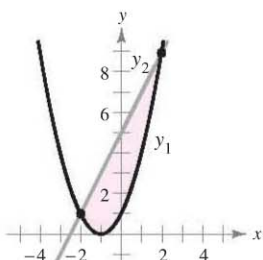

$$A = \int_{-2}^{-2} (4 - x^2) \, dx = 0$$

$$A = \int_{-2}^{-1} (4 - x^2) \, dx = \frac{5}{3}$$

$$A = \int_{-2}^{0} (4 - x^2) \, dx = \frac{16}{3}$$

$$A = \int_{-2}^{1} (4 - x^2) \, dx = 9$$

$$A = \int_{-2}^{2} (4 - x^2) \, dx = \frac{32}{3}$$

Figure 7.11

7.1 | Exercises

In Exercises 1–6, set up the definite integral that gives the area of the region.
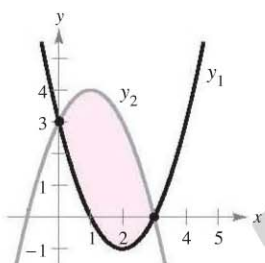
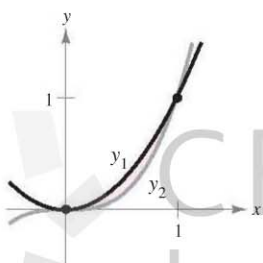**1.** $y_1 = x^2 - 6x$

$y_2 = 0$
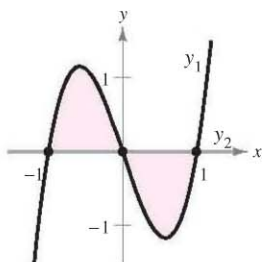

2. $y_1 = x^2 + 2x + 1$

$y_2 = 2x + 5$

3. $y_1 = x^2 - 4x + 3$

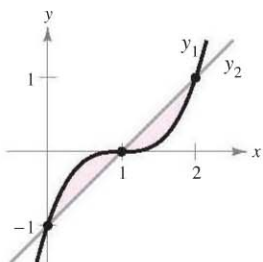$y_2 = -x^2 + 2x + 3$

4. $y_1 = x^2$

$y_2 = x^3$

5. $y_1 = 3(x^3 - x)$

$y_2 = 0$

6. $y_1 = (x - 1)^3$

$y_2 = x - 1$

In Exercises 7–14, the integrand of the definite integral is a difference of two functions. Sketch the graph of each function and shade the region whose area is represented by the integral.

7. $\displaystyle\int_0^4 \left[(x + 1) - \frac{x}{2} \right] dx$

8. $\displaystyle\int_{-1}^1 \left[(2 - x^2) - x^2 \right] dx$

9. $\displaystyle\int_0^6 \left[4(2^{-x/3}) - \frac{x}{6} \right] dx$

10. $\displaystyle\int_2^3 \left[\left(\frac{x^3}{3} - x \right) - \frac{x}{3} \right] dx$

11. $\displaystyle\int_{-\pi/3}^{\pi/3} (2 - \sec x)\, dx$

12. $\displaystyle\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\, dx$

13. $\displaystyle\int_{-2}^1 \left[(2 - y) - y^2 \right] dy$

14. $\displaystyle\int_0^4 \left(2\sqrt{y} - y \right) dy$

Think About It   **In Exercises 15 and 16, determine which value best approximates the area of the region bounded by the graphs of f and g. (Make your selection on the basis of a sketch of the region and not by performing any calculations.)**
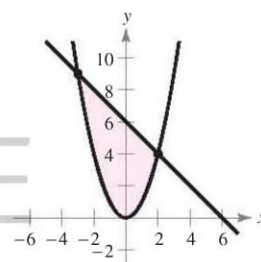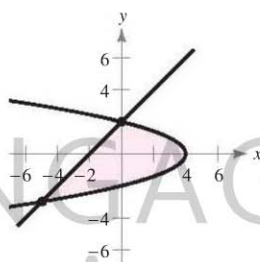
15. $f(x) = x + 1$, $g(x) = (x - 1)^2$

(a) -2 (b) 2   (c) 10   (d) 4   (e) 8

16. $f(x) = 2 - \frac{1}{2}x$, $g(x) = 2 - \sqrt{x}$

(a) 1   (b) 6   (c) -3 (d) 3   (e) 4

In Exercises 17 and 18, find the area of the region by integrating (a) with respect to x and (b) with respect to y. (c) Compare your results. Which method is simpler? In general, will this method always be simpler than the other one? Why or why not?

17. $x = 4 - y^2$

$x = y - 2$

18. $y = x^2$

$y = 6 - x$

In Exercises 19–36, sketch the region bounded by the graphs of the algebraic functions and find the area of the region.

19. $y = x^2 - 1$, $y = -x + 2$, $x = 0$, $x = 1$

20. $y = -x^3 + 3$, $y = x$, $x = -1$, $x = 1$

21. $y = \frac{1}{2}x^3 + 2$, $y = x + 1$, $x = 0$, $x = 2$

22. $y = -\frac{3}{8}x(x - 8)$, $y = 10 - \frac{1}{2}x$, $x = 2$, $x = 8$

23. $f(x) = x^2 - 4x$, $g(x) = 0$

24. $f(x) = -x^2 + 4x + 1$, $g(x) = x + 1$

25. $f(x) = x^2 + 2x$, $g(x) = x + 2$

26. $f(x) = -x^2 + \frac{9}{2}x + 1$, $g(x) = \frac{1}{2}x + 1$

27. $y = x$, $y = 2 - x$, $y = 0$

28. $y = 1/x^2$, $y = 0$, $x = 1$, $x = 5$

29. $f(x) = \sqrt{x} + 3$, $g(x) = \frac{1}{2}x + 3$

30. $f(x) = \sqrt[3]{x - 1}$, $g(x) = x - 1$

31. $f(y) = y^2$, $g(y) = y + 2$

32. $f(y) = y(2 - y)$, $g(y) = -y$

33. $f(y) = y^2 + 1$, $g(y) = 0$, $y = -1$, $y = 2$

34. $f(y) = \dfrac{y}{\sqrt{16 - y^2}}$, $g(y) = 0$, $y = 3$

35. $f(x) = \dfrac{10}{x}$, $x = 0$, $y = 2$, $y = 10$

36. $g(x) = \dfrac{4}{2 - x}$, $y = 4$, $x = 0$

In Exercises 37–46, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region, and (c) use the integration capabilities of the graphing utility to verify your results.

37. $f(x) = x(x^2 - 3x + 3), \ g(x) = x^2$

38. $f(x) = x^3 - 2x + 1, \ g(x) = -2x, \ x = 1$

39. $y = x^2 - 4x + 3, \ y = 3 + 4x - x^2$

40. $y = x^4 - 2x^2, \ y = 2x^2$

41. $f(x) = x^4 - 4x^2, \ g(x) = x^2 - 4$

42. $f(x) = x^4 - 4x^2, \ g(x) = x^3 - 4x$

43. $f(x) = 1/(1 + x^2), \ g(x) = \frac{1}{2}x^2$

44. $f(x) = 6x/(x^2 + 1), \ y = 0, \ 0 \le x \le 3$

45. $y = \sqrt{1 + x^3}, \ y = \frac{1}{2}x + 2, \ x = 0$

46. $y = x\sqrt{\dfrac{4 - x}{4 + x}}, \ y = 0, \ x = 4$

In Exercises 47–52, sketch the region bounded by the graphs of the functions, and find the area of the region.

47. $f(x) = \cos x, \ g(x) = 2 - \cos x, \ 0 \le x \le 2\pi$

48. $f(x) = \sin x, \ g(x) = \cos 2x, \ -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{6}$

49. $f(x) = 2 \sin x, \ g(x) = \tan x, \ -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{3}$

50. $f(x) = \sec\dfrac{\pi x}{4} \tan\dfrac{\pi x}{4}, \ g(x) = (\sqrt{2} - 4)x + 4, \ x = 0$

51. $f(x) = xe^{-x^2}, \ y = 0, \ 0 \le x \le 1$

52. $f(x) = 3^x, \ g(x) = 2x + 1$

In Exercises 53–56, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) find the area of the region, and (c) use the integration capabilities of the graphing utility to verify your results.

53. $f(x) = 2 \sin x + \sin 2x, \ y = 0, \ 0 \le x \le \pi$

54. $f(x) = 2 \sin x + \cos 2x, \ y = 0, \ 0 < x \le \pi$

55. $f(x) = \dfrac{1}{x^2}e^{1/x}, \ y = 0, \ 1 \le x \le 3$

56. $g(x) = \dfrac{4 \ln x}{x}, \ y = 0, \ x = 5$

In Exercises 57–60, (a) use a graphing utility to graph the region bounded by the graphs of the equations, (b) explain why the area of the region is difficult to find by hand, and (c) use the integration capabilities of the graphing utility to approximate the area to four decimal places.

57. $y = \sqrt{\dfrac{x^3}{4 - x}}, \ y = 0, \ x = 3$

58. $y = \sqrt{x}\,e^x, \ y = 0, \ x = 0, \ x = 1$

59. $y = x^2, \ y = 4 \cos x$

60. $y = x^2, \ y = \sqrt{3 + x}$

In Exercises 61–64, find the accumulation function F. Then evaluate F at each value of the independent variable and graphically show the area given by each value of F.

61. $F(x) = \displaystyle\int_0^x \left(\tfrac{1}{2}t + 1\right) dt$ \qquad (a) $F(0)$ \quad (b) $F(2)$ \quad (c) $F(6)$

62. $F(x) = \displaystyle\int_0^x \left(\tfrac{1}{2}t^2 + 2\right) dt$ \qquad (a) $F(0)$ \quad (b) $F(4)$ \quad (c) $F(6)$

63. $F(\alpha) = \displaystyle\int_{-1}^{\alpha} \cos\dfrac{\pi\theta}{2}\, d\theta$ \qquad (a) $F(-1)$ \quad (b) $F(0)$ \quad (c) $F\left(\tfrac{1}{2}\right)$

64. $F(y) = \displaystyle\int_{-1}^{y} 4e^{x/2}\, dx$ \qquad (a) $F(-1)$ \quad (b) $F(0)$ \quad (c) $F(4)$

**In Exercises 65–68, use integration to find the area of the figure having the given vertices.**

65. $(2, -3), (4, 6), (6, 1)$

66. $(0, 0), (a, 0), (b, c)$

67. $(0, 2), (4, 2), (0, -2), (-4, -2)$

68. $(0, 0), (1, 2), (3, -2), (1, -3)$

69. **Numerical Integration** Estimate the surface area of the golf green using (a) the Trapezoidal Rule and (b) Simpson's Rule.
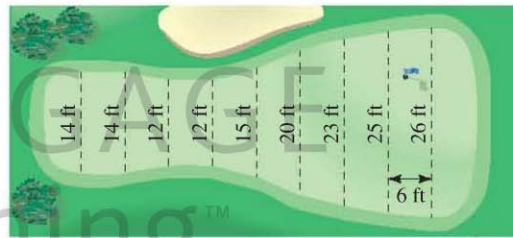
70. **Numerical Integration** Estimate the surface area of the oil spill using (a) the Trapezoidal Rule and (b) Simpson's Rule.
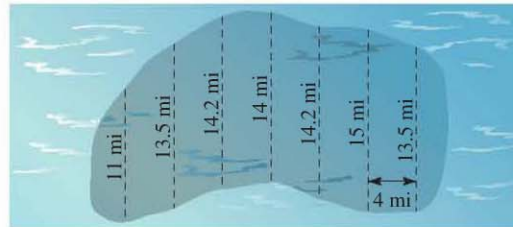
In Exercises 71 and 72, evaluate the integral and interpret it as the area of a region. Then use a graphing utility to graph the region.

71. $\displaystyle\int_0^{\pi/4} |\sin 2x - \cos 4x|\, dx$

72. $\displaystyle\int_0^2 |\sqrt{x + 3} - 2x|\, dx$

In Exercises 73–76, set up and evaluate the definite integral that gives the area of the region bounded by the graph of the function and the tangent line to the graph at the given point.

73. $f(x) = x^3, \ (1, 1)$

74. $y = x^3 - 2x, \ (-1, 1)$

75. $f(x) = \dfrac{1}{x^2 + 1}, \ \left(1, \tfrac{1}{2}\right)$

76. $y = \dfrac{2}{1 + 4x^2}, \ \left(\tfrac{1}{2}, 1\right)$

WRITING ABOUT CONCEPTS

77. The graphs of $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ intersect at three points. However, the area between the curves *can* be found by a single integral. Explain why this is so, and write an integral for this area.

78. The area of the region bounded by the graphs of $y = x^3$ and $y = x$ *cannot* be found by the single integral $\int_{-1}^{1} (x^3 - x)\, dx$. Explain why this is so. Use symmetry to write a single integral that does represent the area.

79. A college graduate has two job offers. The starting salary for each is \$32,000, and after 8 years of service each will pay \$54,000. The salary increase for each offer is shown in the figure. From a strictly monetary viewpoint, which is the better offer? Explain.
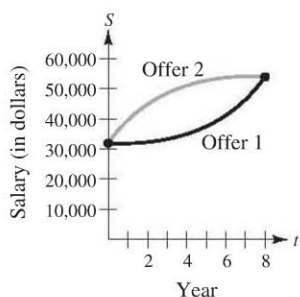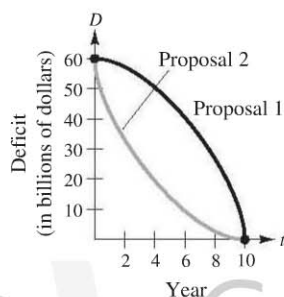

Figure for 79   Figure for 80

80. A state legislature is debating two proposals for eliminating the annual budget deficits after 10 years. The rate of decrease of the deficits for each proposal is shown in the figure. From the viewpoint of minimizing the cumulative state deficit, which is the better proposal? Explain.

81. Two cars are tested on a straight track with velocities v_1 and v_2 (in meters per second). Consider the following.

$$\int_0^5 [v_1(t) - v_2(t)]\, dt = 10 \qquad \int_0^{10} [v_1(t) - v_2(t)]\, dt = 30$$

$$\int_{20}^{30} [v_1(t) - v_2(t)]\, dt = -5$$

(a) Write a verbal interpretation of each integral.

(b) Is it possible to determine the distance between the two cars when $t = 5$ seconds? Why or why not?

(c) Assume both cars start at the same time and place. Which car is ahead when $t = 10$ seconds? How far ahead is the car?

(d) Suppose Car 1 has velocity v_1 and is ahead of Car 2 by 13 meters when $t = 20$ seconds. How far ahead or behind is Car 1 when $t = 30$ seconds?

CAPSTONE

82. Let f and g be continuous functions on $[a, b]$ and let $g(x) \le f(x)$ for all x in $[a, b]$. Write in words the area given by $\int_a^b [f(x) - g(x)]\, dx$. Does the area interpretation of this integral change when $f(x) \ge 0$ and $g(x) \le 0$?

In Exercises 83 and 84, find b such that the line $y = b$ divides the region bounded by the graphs of the two equations into two regions of equal area.

83. $y = 9 - x^2$, $\quad y = 0$ **84.** $y = 9 - |x|$, $\quad y = 0$

In Exercises 85 and 86, find a such that the line $x = a$ divides the region bounded by the graphs of the equations into two regions of equal area.

85. $y = x$, $\quad y = 4$, $\quad x = 0$ **86.** $y^2 = 4 - x$, $\quad x = 0$

In Exercises 87 and 88, evaluate the limit and sketch the graph of the region whose area is represented by the limit.

87. $\displaystyle \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i - x_i^2)\, \Delta x$, where $x_i = i/n$ and $\Delta x = 1/n$

88. $\displaystyle \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - x_i^2)\, \Delta x$, where $x_i = -2 + (4i/n)$ and $\Delta x = 4/n$

In Exercises 89 and 90, (a) find the two points of inflection of the graph of f, (b) determine the equation of the line that intersects both points, and (c) calculate the areas of the three regions bounded by the graphs of f and the line. What do you observe?

89. $f(x) = x^4 + 4x^3 + x + 7$ **90.** $f(x) = 2x^4 - 12x^2 + 3x - 1$

Revenue In Exercises 91 and 92, two models R_1 and R_2 are given for revenue (in billions of dollars per year) for a large corporation. The model R_1 gives projected annual revenues from 2008 through 2013, with $t = 8$ corresponding to 2008, and R_2 gives projected revenues if there is a decrease in the rate of growth of corporate sales over the period. Approximate the total reduction in revenue if corporate sales are actually closer to the model R_2.

91. $R_1 = 7.21 + 0.58t$
$\quad\ R_2 = 7.21 + 0.45t$

92. $R_1 = 7.21 + 0.26t + 0.02t^2$
$\quad\ R_2 = 7.21 + 0.1t + 0.01t^2$

93. *Lorenz Curve* Economists use *Lorenz* curves to illustrate the distribution of income in a country. A Lorenz curve, $y = f(x)$, represents the actual income distribution in the country. In this model, x represents percents of families in the country and y represents percents of total income. The model $y = x$ represents a country in which each family has the same income. The area between these two models, where $0 \le x \le 100$, indicates a country's "income inequality." The table lists percents of income y for selected percents of families x in a country.

x	10	20	30	40	50
y	3.35	6.07	9.17	13.39	19.45

x	60	70	80	90
y	28.03	39.77	55.28	75.12

(a) Use a graphing utility to find a quadratic model for the Lorenz curve.

(b) Plot the data and graph the model.

(c) Graph the model $y = x$. How does this model compare with the model in part (a)?

(d) Use the integration capabilities of a graphing utility to approximate the "income inequality."

94. *Profit* The chief financial officer of a company reports that profits for the past fiscal year were $15.9 million. The officer predicts that profits for the next 5 years will grow at a continuous annual rate somewhere between $3\frac{1}{2}\%$ and 5%. Estimate the cumulative difference in total profit over the 5 years based on the predicted range of growth rates.

95. *Area* The shaded region in the figure consists of all points whose distances from the center of the square are less than their distances from the edges of the square. Find the area of the region.
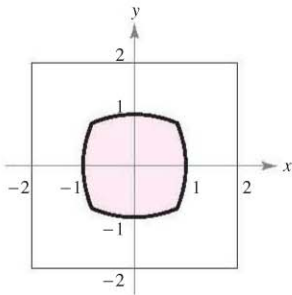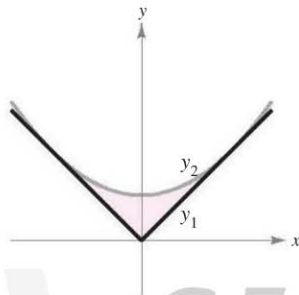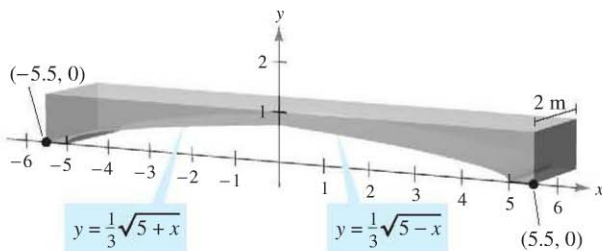

Figure for 95   **Figure for 96**

96. *Mechanical Design* The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$ (see figure).

(a) Find k where the parabola is tangent to the graph of y_1.

(b) Find the area of the surface of the machine part.

97. *Building Design* Concrete sections for a new building have the dimensions (in meters) and shape shown in the figure.

(a) Find the area of the face of the section superimposed on the rectangular coordinate system.

(b) Find the volume of concrete in one of the sections by multiplying the area in part (a) by 2 meters.

(c) One cubic meter of concrete weighs 5000 pounds. Find the weight of the section.

98. *Building Design* To decrease the weight and to aid in the hardening process, the concrete sections in Exercise 97 often are not completely solid. Rework Exercise 97 to allow for cylindrical openings such as those shown in the figure.
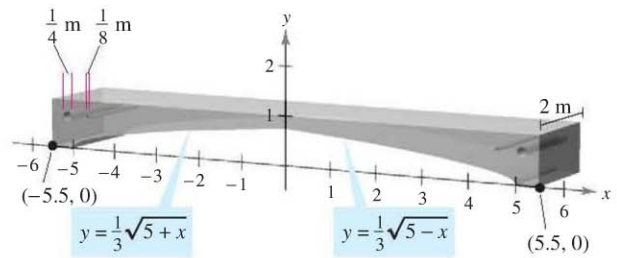

Figure for 98

True or False? In Exercises 99–102, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

99. If the area of the region bounded by the graphs of f and g is 1, then the area of the region bounded by the graphs of $h(x) = f(x) + C$ and $k(x) = g(x) + C$ is also 1.

100. If $\int_a^b [f(x) - g(x)]\, dx = A$, then $\int_a^b [g(x) - f(x)]\, dx = -A$.

101. If the graphs of f and g intersect midway between $x = a$ and $x = b$, then $\int_a^b [f(x) - g(x)]\, dx = 0$.

102. The line $y = \left(1 - \sqrt[3]{0.5}\right)x$ divides the region under the curve $f(x) = x(1 - x)$ on $[0, 1]$ into two regions of equal area.

103. *Area* Find the area between the graph of $y = \sin x$ and the line segment joining the points $(0, 0)$ and $\left(\dfrac{7\pi}{6}, -\dfrac{1}{2}\right)$, as shown in the figure.
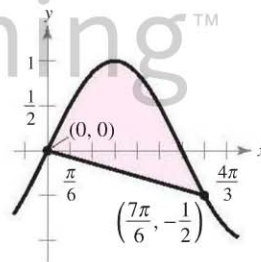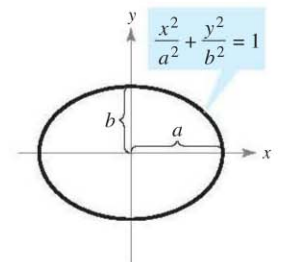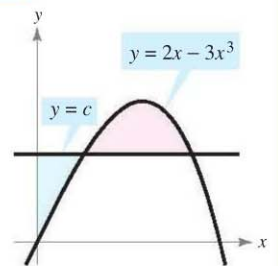

Figure for 103   **Figure for 104**

104. *Area* Let $a > 0$ and $b > 0$. Show that the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\pi a b$ (see figure).

PUTNAM EXAM CHALLENGE

105. The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as shown in the figure. Find c so that the areas of the two shaded regions are equal.

This problem was composed by the Committee on the Putnam Prize Competition.
© The Mathematical Association of America. All rights reserved.