2.4 The Chain Rule

- **Find the derivative of a composite function using the Chain Rule.**
- **Find the derivative of a function using the General Power Rule.**
- **Simplify the derivative of a function using algebra.**
- **Find the derivative of a trigonometric function using the Chain Rule.**

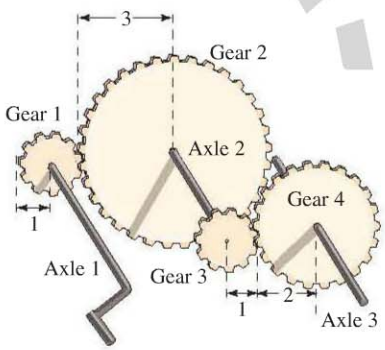
The Chain Rule

This text has yet to discuss one of the most powerful differentiation rules—the **Chain Rule.** This rule deals with composite functions and adds a surprising versatility to the rules discussed in the two previous sections. For example, compare the functions shown below. Those on the left can be differentiated without the Chain Rule, and those on the right are best differentiated with the Chain Rule.

Without the Chain Rule	*With the Chain Rule*
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$
$y = x + \tan x$	$y = x + \tan x^2$

Basically, the Chain Rule states that if y changes dy/du times as fast as u, and u changes du/dx times as fast as x, then y changes $(dy/du)(du/dx)$ times as fast as x.

EXAMPLE 1 The Derivative of a Composite Function



Axle 1: y revolutions per minute
Axle 2: u revolutions per minute
Axle 3: x revolutions per minute
Figure 2.24

A set of gears is constructed, as shown in Figure 2.24, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles, respectively. Find dy/du, du/dx, and dy/dx, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Solution Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. So, you can write

$$\frac{dy}{dx} = \boxed{\begin{array}{l}\text{Rate of change of first axle}\\ \text{with respect to second axle}\end{array}} \cdot \boxed{\begin{array}{l}\text{Rate of change of second axle}\\ \text{with respect to third axle}\end{array}}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6$$

$$= \boxed{\begin{array}{l}\text{Rate of change of first axle}\\ \text{with respect to third axle}\end{array}}.$$

In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x. ■

EXPLORATION

Using the Chain Rule Each of the following functions can be differentiated using rules that you studied in Sections 2.2 and 2.3. For each function, find the derivative using those rules. Then find the derivative using the Chain Rule. Compare your results. Which method is simpler?

a. $\dfrac{2}{3x + 1}$

b. $(x + 2)^3$

c. $\sin 2x$

Example 1 illustrates a simple case of the Chain Rule. The general rule is stated below.

THEOREM 2.10 THE CHAIN RULE

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

PROOF Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c. A problem occurs if there are values of x, other than c, such that $g(x) = g(c)$. Appendix A shows how to use the differentiability of f and g to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c. In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because g is differentiable, it is also continuous, and it follows that $g(x) \rightarrow g(c)$ as $x \rightarrow c$.

$$\begin{aligned} h'(c) &= \lim_{x\to c} \frac{f(g(x)) - f(g(c))}{x - c} \\ &= \lim_{x\to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}\right], \quad g(x) \neq g(c) \\ &= \left[\lim_{x\to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)}\right]\left[\lim_{x\to c} \frac{g(x) - g(c)}{x - c}\right] \\ &= f'(g(c))g'(c) \end{aligned}$$

■

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

EXAMPLE 2 **Decomposition of a Composite Function**

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \dfrac{1}{x + 1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b. $y = \sin 2x$	$u = 2x$	$y = \sin u$
c. $y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d. $y = \tan^2 x$	$u = \tan x$	$y = u^2$

EXAMPLE 3 **Using the Chain Rule**

Find dy/dx for $y = (x^2 + 1)^3$.

Solution For this function, you can consider the inside function to be $u = x^2 + 1$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2 + 1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2 + 1)^2.$$

■

STUDY TIP You could also solve the problem in Example 3 without using the Chain Rule by observing that

$$y = x^6 + 3x^4 + 3x^2 + 1$$

and

$$y' = 6x^5 + 12x^3 + 6x.$$

Verify that this is the same as the derivative in Example 3. Which method would you use to find

$$\frac{d}{dx}(x^2 + 1)^{50}?$$

The General Power Rule

The function in Example 3 is an example of one of the most common types of composite functions, $y = [u(x)]^n$. The rule for differentiating such functions is called the **General Power Rule,** and it is a special case of the Chain Rule.

THEOREM 2.11 THE GENERAL POWER RULE

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = n\,u^{n-1}\,u'.$$

PROOF Because $y = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$
$$= \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (Simple) Power Rule in Section 2.2, you have $D_u[u^n] = n\,u^{n-1}$, and it follows that

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}.$$

■

EXAMPLE 4 Applying the General Power Rule

Find the derivative of $f(x) = (3x - 2x^2)^3$.

Solution Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

$$\begin{aligned} f'(x) &= \overset{n}{3}\overbrace{(3x - 2x^2)^2}^{u^{n-1}}\overbrace{\frac{d}{dx}[3x - 2x^2]}^{u'} && \text{Apply General Power Rule.} \\ &= 3(3x - 2x^2)^2(3 - 4x). && \text{Differentiate } 3x - 2x^2. \end{aligned}$$

EXAMPLE 5 Differentiating Functions Involving Radicals

Find all points on the graph of $f(x) = \sqrt[3]{(x^2 - 1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

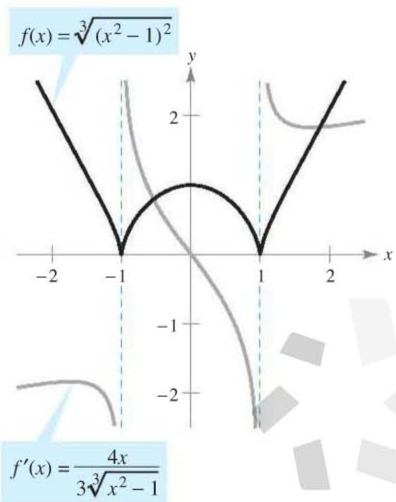
Solution Begin by rewriting the function as

$$f(x) = (x^2 - 1)^{2/3}.$$

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

$$\begin{aligned} f'(x) &= \overset{n}{\frac{2}{3}}\overbrace{(x^2 - 1)^{-1/3}}^{u^{n-1}}\overbrace{(2x)}^{u'} && \text{Apply General Power Rule.} \\ &= \frac{4x}{3\sqrt[3]{x^2 - 1}}. && \text{Write in radical form.} \end{aligned}$$

So, $f'(x) = 0$ when $x = 0$ and $f'(x)$ does not exist when $x = \pm 1$, as shown in Figure 2.25.



The derivative of f is 0 at $x = 0$ and is undefined at $x = \pm 1$.
Figure 2.25

EXAMPLE 6 Differentiating Quotients with Constant Numerators

Differentiate $g(t) = \dfrac{-7}{(2t - 3)^2}$.

Solution Begin by rewriting the function as

$$g(t) = -7(2t - 3)^{-2}.$$

Then, applying the General Power Rule produces

$$\begin{aligned} g'(t) &= \underbrace{(-7)}_{\text{Constant Multiple Rule}}\overbrace{(-2)}^{n}\overbrace{(2t - 3)^{-3}}^{u^{n-1}}\overbrace{(2)}^{u'} && \text{Apply General Power Rule.} \\ &= 28(2t - 3)^{-3} && \text{Simplify.} \\ &= \frac{28}{(2t - 3)^3}. && \text{Write with positive exponent.} \end{aligned}$$

■

NOTE Try differentiating the function in Example 6 using the Quotient Rule. You should obtain the same result, but using the Quotient Rule is less efficient than using the General Power Rule.

Simplifying Derivatives

The next three examples illustrate some techniques for simplifying the "raw derivatives" of functions involving products, quotients, and composites.

EXAMPLE 7 Simplifying by Factoring Out the Least Powers

$$f(x) = x^2\sqrt{1-x^2}$$ Original function

$$= x^2(1-x^2)^{1/2}$$ Rewrite.

$$f'(x) = x^2\frac{d}{dx}[(1-x^2)^{1/2}] + (1-x^2)^{1/2}\frac{d}{dx}[x^2]$$ Product Rule

$$= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x)$$ General Power Rule

$$= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2}$$ Simplify.

$$= x(1-x^2)^{-1/2}[-x^2(1) + 2(1-x^2)]$$ Factor.

$$= \frac{x(2-3x^2)}{\sqrt{1-x^2}}$$ Simplify.

EXAMPLE 8 Simplifying the Derivative of a Quotient

TECHNOLOGY Symbolic differentiation utilities are capable of differentiating very complicated functions. Often, however, the result is given in unsimplified form. If you have access to such a utility, use it to find the derivatives of the functions given in Examples 7, 8, and 9. Then compare the results with those given in these examples.

$$f(x) = \frac{x}{\sqrt[3]{x^2+4}}$$ Original function

$$= \frac{x}{(x^2+4)^{1/3}}$$ Rewrite.

$$f'(x) = \frac{(x^2+4)^{1/3}(1) - x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}}$$ Quotient Rule

$$= \frac{1}{3}(x^2+4)^{-2/3}\left[\frac{3(x^2+4) - (2x^2)(1)}{(x^2+4)^{2/3}}\right]$$ Factor.

$$= \frac{x^2+12}{3(x^2+4)^{4/3}}$$ Simplify.

EXAMPLE 9 Simplifying the Derivative of a Power

$$y = \left(\frac{3x-1}{x^2+3}\right)^2$$ Original function

$$y' = \underbrace{2}_{n}\underbrace{\left(\frac{3x-1}{x^2+3}\right)}_{u^{n-1}}\underbrace{\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]}_{u'}$$ General Power Rule

$$= \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right]$$ Quotient Rule

$$= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3}$$ Multiply.

$$= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}$$ Simplify. ■

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

EXAMPLE 10 Applying the Chain Rule to Trigonometric Functions

a. $y = \sin \underbrace{2x}_{u}$ $\quad y' = \underbrace{\cos 2x}_{\cos u}\underbrace{\frac{d}{dx}[2x]}_{u'} = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos(x - 1)$ $\quad y' = -\sin(x - 1)$

c. $y = \tan 3x$ $\quad y' = 3\sec^2 3x$ ■

Be sure that you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10(a), $\sin 2x$ is written to mean $\sin(2x)$.

EXAMPLE 11 Parentheses and Trigonometric Functions

a. $y = \cos 3x^2 = \cos(3x^2)$ $\quad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

b. $y = (\cos 3)x^2$ $\quad y' = (\cos 3)(2x) = 2x \cos 3$

c. $y = \cos(3x)^2 = \cos(9x^2)$ $\quad y' = (-\sin 9x^2)(18x) = -18x \sin 9x^2$

d. $y = \cos^2 x = (\cos x)^2$ $\quad y' = 2(\cos x)(-\sin x) = -2\cos x \sin x$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$ $\quad y' = \frac{1}{2}(\cos x)^{-1/2}(-\sin x) = -\frac{\sin x}{2\sqrt{\cos x}}$

■

To find the derivative of a function of the form $k(x) = f(g(h(x)))$, you need to apply the Chain Rule twice, as shown in Example 12.

EXAMPLE 12 Repeated Application of the Chain Rule

$f(t) = \sin^3 4t$ — Original function

$= (\sin 4t)^3$ — Rewrite.

$f'(t) = 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t]$ — Apply Chain Rule once.

$= 3(\sin 4t)^2(\cos 4t)\frac{d}{dt}[4t]$ — Apply Chain Rule a second time.

$= 3(\sin 4t)^2(\cos 4t)(4)$

$= 12\sin^2 4t \cos 4t$ — Simplify. ■

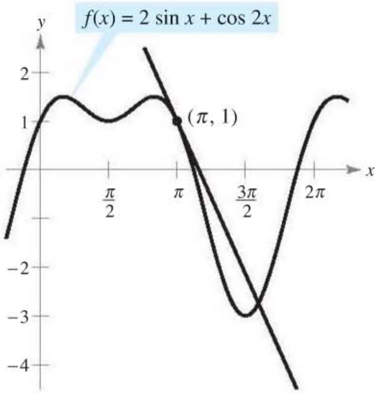


Figure 2.26

EXAMPLE 13 Tangent Line of a Trigonometric Function

Find an equation of the tangent line to the graph of

$$f(x) = 2\sin x + \cos 2x$$

at the point $(\pi, 1)$, as shown in Figure 2.26. Then determine all values of x in the interval $(0, 2\pi)$ at which the graph of f has a horizontal tangent.

Solution Begin by finding $f'(x)$.

$$f(x) = 2\sin x + \cos 2x \qquad \text{Write original function.}$$
$$f'(x) = 2\cos x + (-\sin 2x)(2) \qquad \text{Apply Chain Rule to } \cos 2x.$$
$$= 2\cos x - 2\sin 2x \qquad \text{Simplify.}$$

To find the equation of the tangent line at $(\pi, 1)$, evaluate $f'(\pi)$.

$$f'(\pi) = 2\cos \pi - 2\sin 2\pi \qquad \text{Substitute.}$$
$$= -2 \qquad \text{Slope of graph at } (\pi, 1)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$
$$y - 1 = -2(x - \pi) \qquad \text{Substitute for } y_1, m, \text{ and } x_1.$$
$$y = 1 - 2x + 2\pi. \qquad \text{Equation of tangent line at } (\pi, 1)$$

You can then determine that $f'(x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$. So, f has horizontal tangents at $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$. ■

STUDY TIP To become skilled at differentiation, you should memorize each rule in words, not symbols. As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

This section concludes with a summary of the differentiation rules studied so far.

SUMMARY OF DIFFERENTIATION RULES

General Differentiation Rules Let f, g, and u be differentiable functions of x.

Constant Multiple Rule: $\frac{d}{dx}[cf] = cf'$

Sum or Difference Rule: $\frac{d}{dx}[f \pm g] = f' \pm g'$

Product Rule: $\frac{d}{dx}[fg] = fg' + gf'$

Quotient Rule: $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$

Derivatives of Algebraic Functions

Constant Rule: $\frac{d}{dx}[c] = 0$

(Simple) Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$

Derivatives of Trigonometric Functions

$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$

Chain Rule

Chain Rule: $\frac{d}{dx}[f(u)] = f'(u)\, u'$

General Power Rule: $\frac{d}{dx}[u^n] = nu^{n-1}\, u'$

2.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, complete the table.

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (5x - 8)^4$		
2. $y = \dfrac{1}{\sqrt{x+1}}$		
3. $y = \sqrt{x^3 - 7}$		
4. $y = 3\tan(\pi x^2)$		
5. $y = \csc^3 x$		
6. $y = \sin\dfrac{5x}{2}$		

In Exercises 7–36, find the derivative of the function.

7. $y = (4x - 1)^3$

8. $y = 2(6 - x^2)^5$

9. $g(x) = 3(4 - 9x)^4$

10. $f(t) = (9t + 2)^{2/3}$

11. $f(t) = \sqrt{5 - t}$

12. $g(x) = \sqrt{9 - 4x}$

13. $y = \sqrt[3]{6x^2 + 1}$

14. $g(x) = \sqrt{x^2 - 2x + 1}$

15. $y = 2\sqrt[4]{9 - x^2}$

16. $f(x) = -3\sqrt[4]{2 - 9x}$

17. $y = \dfrac{1}{x - 2}$

18. $s(t) = \dfrac{1}{t^2 + 3t - 1}$

19. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$

20. $y = -\dfrac{5}{(t + 3)^3}$

21. $y = \dfrac{1}{\sqrt{x + 2}}$

22. $g(t) = \sqrt{\dfrac{1}{t^2 - 2}}$

23. $f(x) = x^2(x - 2)^4$

24. $f(x) = x(3x - 9)^3$

25. $y = x\sqrt{1 - x^2}$

26. $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

27. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

28. $y = \dfrac{x}{\sqrt{x^4 + 4}}$

29. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

30. $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

31. $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

32. $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

33. $f(x) = ((x^2 + 3)^5 + x)^2$

34. $g(x) = (2 + (x^2 + 1)^4)^3$

35. $f(x) = \sqrt{2 + \sqrt{2 + \sqrt{x}}}$

36. $g(t) = \sqrt{\sqrt{t + 1} + 1}$

CAS In Exercises 37–42, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

37. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

38. $y = \sqrt{\dfrac{2x}{x + 1}}$

39. $y = \sqrt{\dfrac{x + 1}{x}}$

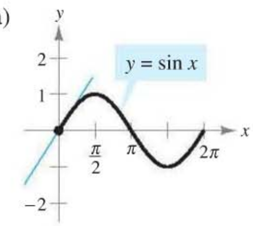
40. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

41. $y = \dfrac{\cos \pi x + 1}{x}$

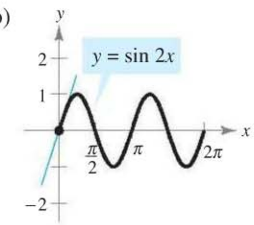
42. $y = x^2 \tan\dfrac{1}{x}$

In Exercises 43 and 44, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

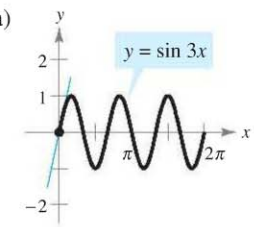
43. (a)



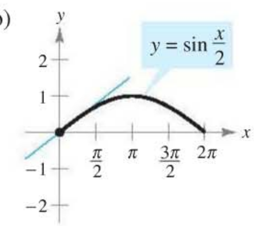
(b)



44. (a)



(b)



In Exercises 45–66, find the derivative of the function.

45. $y = \cos 4x$

46. $y = \sin \pi x$

47. $g(x) = 5\tan 3x$

48. $h(x) = \sec x^2$

49. $y = \sin(\pi x)^2$

50. $y = \cos(1 - 2x)^2$

51. $h(x) = \sin 2x \cos 2x$

52. $g(\theta) = \sec(\frac{1}{2}\theta)\tan(\frac{1}{2}\theta)$

53. $f(x) = \dfrac{\cot x}{\sin x}$

54. $g(v) = \dfrac{\cos v}{\csc v}$

55. $y = 4\sec^2 x$

56. $g(t) = 5\cos^2 \pi t$

57. $f(\theta) = \tan^2 5\theta$

58. $g(\theta) = \cos^2 8\theta$

59. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

60. $h(t) = 2\cot^2(\pi t + 2)$

61. $f(t) = 3\sec^2(\pi t - 1)$

62. $y = 3x - 5\cos(\pi x)^2$

63. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

64. $y = \sin\sqrt[3]{x} + \sqrt[3]{\sin x}$

65. $y = \sin(\tan 2x)$

66. $y = \cos\sqrt{\sin(\tan \pi x)}$

In Exercises 67–74, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

67. $s(t) = \sqrt{t^2 + 6t - 2}$, $(3, 5)$

68. $y = \sqrt[5]{3x^3 + 4x}$, $(2, 2)$

69. $f(x) = \dfrac{5}{x^3 - 2}$, $\left(-2, -\dfrac{1}{2}\right)$

70. $f(x) = \dfrac{1}{(x^2 - 3x)^2}$, $\left(4, \dfrac{1}{16}\right)$

71. $f(t) = \dfrac{3t + 2}{t - 1}$, $(0, -2)$

72. $f(x) = \dfrac{x + 1}{2x - 3}$, $(2, 3)$

73. $y = 26 - \sec^3 4x$, $(0, 25)$

74. $y = \dfrac{1}{x} + \sqrt{\cos x}$, $\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

In Exercises 75–82, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.

Function	*Point*
75. $f(x) = \sqrt{2x^2 - 7}$	$(4, 5)$
76. $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$	$(2, 2)$
77. $y = (4x^3 + 3)^2$	$(-1, 1)$
78. $f(x) = (9 - x^2)^{2/3}$	$(1, 4)$
79. $f(x) = \sin 2x$	$(\pi, 0)$
80. $y = \cos 3x$	$\left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
81. $f(x) = \tan^2 x$	$\left(\frac{\pi}{4}, 1\right)$
82. $y = 2\tan^3 x$	$\left(\frac{\pi}{4}, 2\right)$

In Exercises 83–86, (a) use a graphing utility to find the derivative of the function at the given point, (b) find an equation of the tangent line to the graph of the function at the given point, and (c) use the utility to graph the function and its tangent line in the same viewing window.

83. $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}, \quad \left(\frac{1}{2}, \frac{3}{2}\right)$

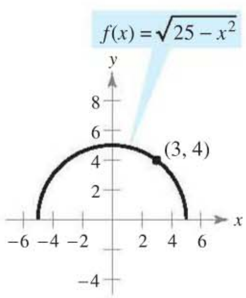
84. $f(x) = \sqrt{x}(2 - x)^2, \quad (4, 8)$

85. $s(t) = \dfrac{(4 - 2t)\sqrt{1 + t}}{3}, \quad \left(0, \frac{4}{3}\right)$

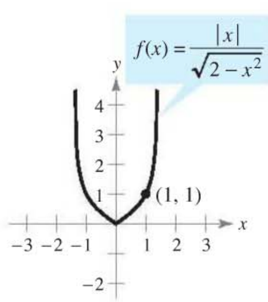
86. $y = (t^2 - 9)\sqrt{t + 2}, \quad (2, -10)$

Famous Curves **In Exercises 87 and 88, find an equation of the tangent line to the graph at the given point. Then use a graphing utility to graph the function and its tangent line in the same viewing window.**

87. Top half of circle



88. Bullet-nose curve



89. ***Horizontal Tangent Line*** Determine the point(s) in the interval $(0, 2\pi)$ at which the graph of $f(x) = 2\cos x + \sin 2x$ has a horizontal tangent.

90. ***Horizontal Tangent Line*** Determine the point(s) at which the graph of $f(x) = \dfrac{x}{\sqrt{2x - 1}}$ has a horizontal tangent.

In Exercises 91–96, find the second derivative of the function.

91. $f(x) = 5(2 - 7x)^4$

92. $f(x) = 4(x^2 - 2)^3$

93. $f(x) = \dfrac{1}{x - 6}$

94. $f(x) = \dfrac{4}{(x + 2)^3}$

95. $f(x) = \sin x^2$

96. $f(x) = \sec^2 \pi x$

In Exercises 97–100, evaluate the second derivative of the function at the given point. Use a computer algebra system to verify your result.

97. $h(x) = \frac{1}{9}(3x + 1)^3, \quad \left(1, \frac{64}{9}\right)$

98. $f(x) = \dfrac{1}{\sqrt{x + 4}}, \quad \left(0, \frac{1}{2}\right)$

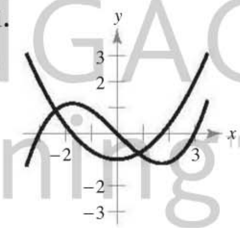
99. $f(x) = \cos x^2, \quad (0, 1)$

100. $g(t) = \tan 2t, \quad \left(\frac{\pi}{6}, \sqrt{3}\right)$

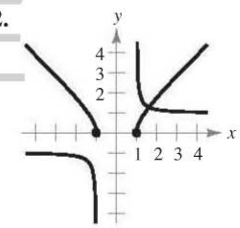
WRITING ABOUT CONCEPTS

In Exercises 101–104, the graphs of a function f and its derivative f' are shown. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

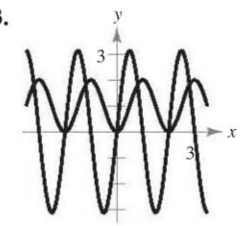
101.



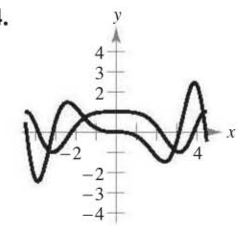
102.



103.



104.



In Exercises 105 and 106, the relationship between f and g is given. Explain the relationship between f' and g'.

105. $g(x) = f(3x)$

106. $g(x) = f(x^2)$

107. ***Think About It*** The table shows some values of the derivative of an unknown function f. Complete the table by finding (if possible) the derivative of each transformation of f.

(a) $g(x) = f(x) - 2$

(b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$

(d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

Table for 107

CAPSTONE

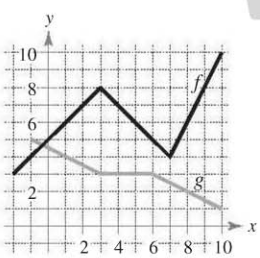
108. Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$ (b) $f(x) = g(h(x))$

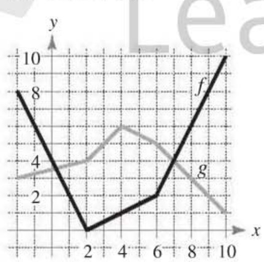
(c) $f(x) = \dfrac{g(x)}{h(x)}$ (d) $f(x) = [g(x)]^3$

In Exercises 109 and 110, the graphs of f and g are shown. Let $h(x) = f(g(x))$ and $s(x) = g(f(x))$. Find each derivative, if it exists. If the derivative does not exist, explain why.

109. (a) Find $h'(1)$.
(b) Find $s'(5)$.



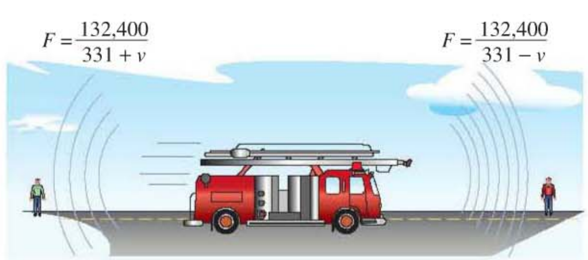
110. (a) Find $h'(3)$.
(b) Find $s'(9)$.



111. ***Doppler Effect*** The frequency F of a fire truck siren heard by a stationary observer is $F = 132{,}400/(331 \pm v)$, where $\pm v$ represents the velocity of the accelerating fire truck in meters per second (see figure). Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

(b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).



112. ***Harmonic Motion*** The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

113. ***Pendulum*** A 15-centimeter pendulum moves according to the equation $\theta = 0.2\cos 8t$, where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

114. ***Wave Motion*** A buoy oscillates in simple harmonic motion $y = A\cos \omega t$ as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

115. ***Circulatory System*** The speed S of blood that is r centimeters from the center of an artery is

$$S = C(R^2 - r^2)$$

where C is a constant, R is the radius of the artery, and S is measured in centimeters per second. Suppose a drug is administered and the artery begins to dilate at a rate of dR/dt. At a constant distance r, find the rate at which S changes with respect to t for $C = 1.76 \times 10^5$, $R = 1.2 \times 10^{-2}$, and $dR/dt = 10^{-5}$.

116. ***Modeling Data*** The normal daily maximum temperatures T (in degrees Fahrenheit) for Chicago, Illinois are shown in the table. *(Source: National Oceanic and Atmospheric Administration)*

Month	Jan	Feb	Mar	Apr	May	Jun
Temperature	29.6	34.7	46.1	58.0	69.9	79.2

Month	Jul	Aug	Sep	Oct	Nov	Dec
Temperature	83.5	81.2	73.9	62.1	47.1	34.4

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b\sin(ct - d)$$

where T is the temperature and t is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of the temperature changes? Explain..5