# 1.1 A Preview of Calculus

- **Understand what calculus is and how it compares with precalculus.**
- **Understand that the tangent line problem is basic to calculus.**
- **Understand that the area problem is also basic to calculus.**

## What Is Calculus?

Calculus is the mathematics of change. For instance, calculus is the mathematics of velocities, accelerations, tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

Although precalculus mathematics also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus. Precalculus mathematics is more static, whereas calculus is more dynamic. Here are some examples.

- An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.
- The curvature of a circle is constant and can be analyzed with precalculus mathematics. To analyze the variable curvature of a general curve, you need calculus.
- The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.

Each of these situations involves the same general strategy—the reformulation of precalculus mathematics through the use of a limit process. So, one way to answer the question "What is calculus?" is to say that calculus is a "limit machine" that involves three stages. The first stage is precalculus mathematics, such as the slope of a line or the area of a rectangle. The second stage is the limit process, and the third stage is a new calculus formulation, such as a derivative or integral.

Precalculus mathematics ⇨ Limit process ⇨ Calculus

Some students try to learn calculus as if it were simply a collection of new formulas. This is unfortunate. If you reduce calculus to the memorization of differentiation and integration formulas, you will miss a great deal of understanding, self-confidence, and satisfaction.

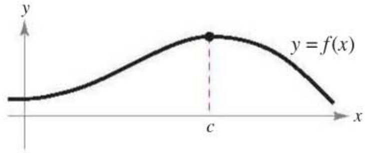
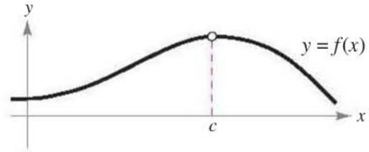
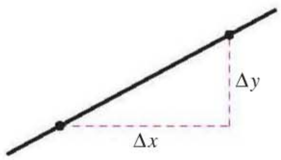
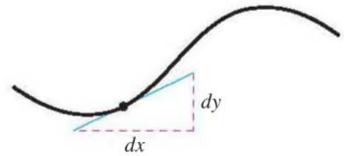
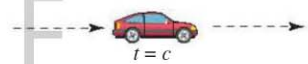
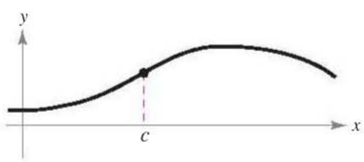
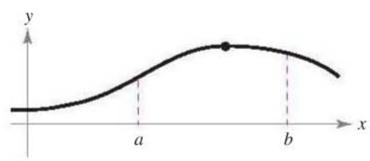
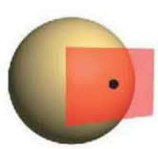
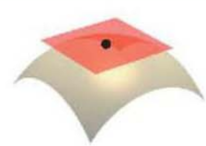
On the following two pages are listed some familiar precalculus concepts coupled with their calculus counterparts. Throughout the text, your goal should be to learn how precalculus formulas and techniques are used as building blocks to produce the more general calculus formulas and techniques. Don't worry if you are unfamiliar with some of the "old formulas" listed on the following two pages—you will be reviewing all of them.

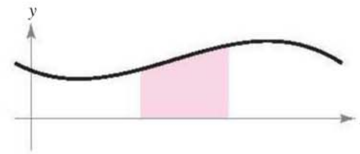
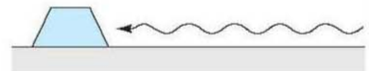
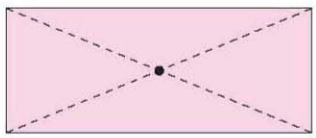
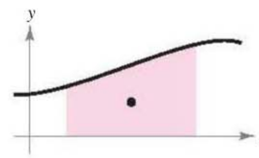
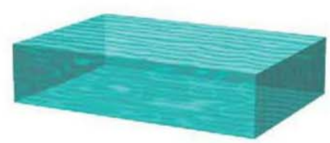
As you proceed through this text, come back to this discussion repeatedly. Try to keep track of where you are relative to the three stages involved in the study of calculus. For example, the first three chapters break down as follows.

| | |
|---|---|
| Chapter P: Preparation for Calculus | Precalculus |
| Chapter 1: Limits and Their Properties | Limit process |
| Chapter 2: Differentiation | Calculus |

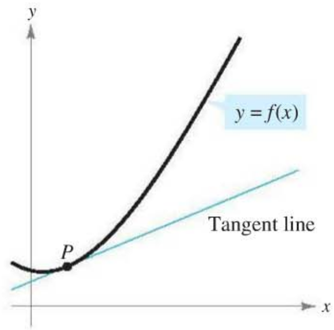
**STUDY TIP** As you progress through this course, remember that learning calculus is just one of your goals. Your most important goal is to learn how to use calculus to model and solve real-life problems. Here are a few problem-solving strategies that may help you.

- Be sure you understand the question. What is given? What are you asked to find?
- Outline a plan. There are many approaches you could use: look for a pattern, solve a simpler problem, work backwards, draw a diagram, use technology, or any of many other approaches.
- Complete your plan. Be sure to answer the question. Verbalize your answer. For example, rather than writing the answer as $x = 4.6$, it would be better to write the answer as "The area of the region is 4.6 square meters."
- Look back at your work. Does your answer make sense? Is there a way you can check the reasonableness of your answer?

| Without Calculus | | With Differential Calculus | |
|---|---|---|---|
| Value of $f(x)$ when $x = c$ |   | Limit of $f(x)$ as $x$ approaches $c$ |   |
| Slope of a line |   | Slope of a curve |   |
| Secant line to a curve |  | Tangent line to a curve |  |
| Average rate of change between $t = a$ and $t = b$ |   | Instantaneous rate of change at $t = c$ |   |
| Curvature of a circle |  | Curvature of a curve |  |
| Height of a curve when $x = c$ |   | Maximum height of a curve on an interval |   |
| Tangent plane to a sphere |  | Tangent plane to a surface |  |
| Direction of motion along a line |  | Direction of motion along a curve |  |

| Without Calculus | | With Integral Calculus | |
|---|---|---|---|
| Area of a rectangle |  | Area under a curve |  |
| Work done by a constant force |  | Work done by a variable force |  |
| Center of a rectangle |  | Centroid of a region |  |
| Length of a line segment |  | Length of an arc |  |
| Surface area of a cylinder |  | Surface area of a solid of revolution |  |
| Mass of a solid of constant density |  | Mass of a solid of variable density |  |
| Volume of a rectangular solid |  | Volume of a region under a surface |  |
| Sum of a finite number of terms | $a_1 + a_2 + \cdot \cdot \cdot + a_n = S$ | Sum of an infinite number of terms | $a_1 + a_2 + a_3 + \cdot \cdot \cdot = S$ |

## The Tangent Line Problem




The tangent line to the graph of $f$ at $P$
**Figure 1.1**

The notion of a limit is fundamental to the study of calculus. The following brief descriptions of two classic problems in calculus—*the tangent line problem* and *the area problem*—should give you some idea of the way limits are used in calculus.

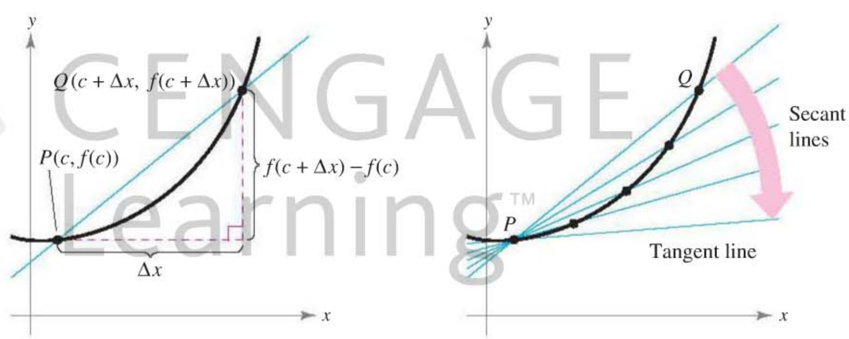
In the tangent line problem, you are given a function $f$ and a point $P$ on its graph and are asked to find an equation of the tangent line to the graph at point $P$, as shown in Figure 1.1.

Except for cases involving a vertical tangent line, the problem of finding the **tangent line** at a point $P$ is equivalent to finding the *slope* of the tangent line at $P$. You can approximate this slope by using a line through the point of tangency and a second point on the curve, as shown in Figure 1.2(a). Such a line is called a **secant line.** If $P(c, f(c))$ is the point of tangency and

$$Q(c + \Delta x, f(c + \Delta x))$$

is a second point on the graph of $f$, the slope of the secant line through these two points can be found using precalculus and is given by

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$




**(a)** The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

**(b)** As $Q$ approaches $P$, the secant lines approach the tangent line.

**Figure 1.2**

As point $Q$ approaches point $P$, the slopes of the secant lines approach the slope of the tangent line, as shown in Figure 1.2(b). When such a "limiting position" exists, the slope of the tangent line is said to be the **limit** of the slopes of the secant lines. (Much more will be said about this important calculus concept in Chapter 2.)

The Mistress Fellows, Girton College, Cambridge

**GRACE CHISHOLM YOUNG (1868–1944)**

Grace Chisholm Young received her degree in mathematics from Girton College in Cambridge, England. Her early work was published under the name of William Young, her husband. Between 1914 and 1916, Grace Young published work on the foundations of calculus that won her the Gamble Prize from Girton College.

**EXPLORATION**

The following points lie on the graph of $f(x) = x^2$.

$Q_1(1.5, f(1.5))$, $Q_2(1.1, f(1.1))$, $Q_3(1.01, f(1.01))$,

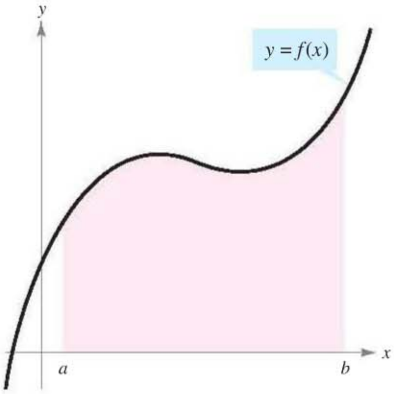
$Q_4(1.001, f(1.001))$, $Q_5(1.0001, f(1.0001))$

Each successive point gets closer to the point $P(1, 1)$. Find the slopes of the secant lines through $Q_1$ and $P$, $Q_2$ and $P$, and so on. Graph these secant lines on a graphing utility. Then use your results to estimate the slope of the tangent line to the graph of $f$ at the point $P$.

## The Area Problem

In the tangent line problem, you saw how the limit process can be applied to the slope of a line to find the slope of a general curve. A second classic problem in calculus is finding the area of a plane region that is bounded by the graphs of functions. This problem can also be solved with a limit process. In this case, the limit process is applied to the area of a rectangle to find the area of a general region.

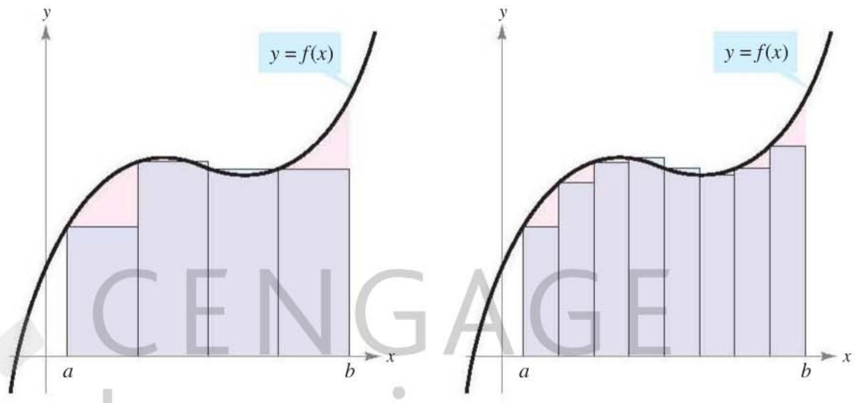
As a simple example, consider the region bounded by the graph of the function $y = f(x)$, the $x$-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 1.3. You can approximate the area of the region with several rectangular regions, as shown in Figure 1.4. As you increase the number of rectangles, the approximation tends to become better and better because the amount of area missed by the rectangles decreases. Your goal is to determine the limit of the sum of the areas of the rectangles as the number of rectangles increases without bound.




Area under a curve
**Figure 1.3**

**HISTORICAL NOTE**

In one of the most astounding events ever to occur in mathematics, it was discovered that the tangent line problem and the area problem are closely related. This discovery led to the birth of calculus. You will learn about the relationship between these two problems when you study the Fundamental Theorem of Calculus in Chapter 4.

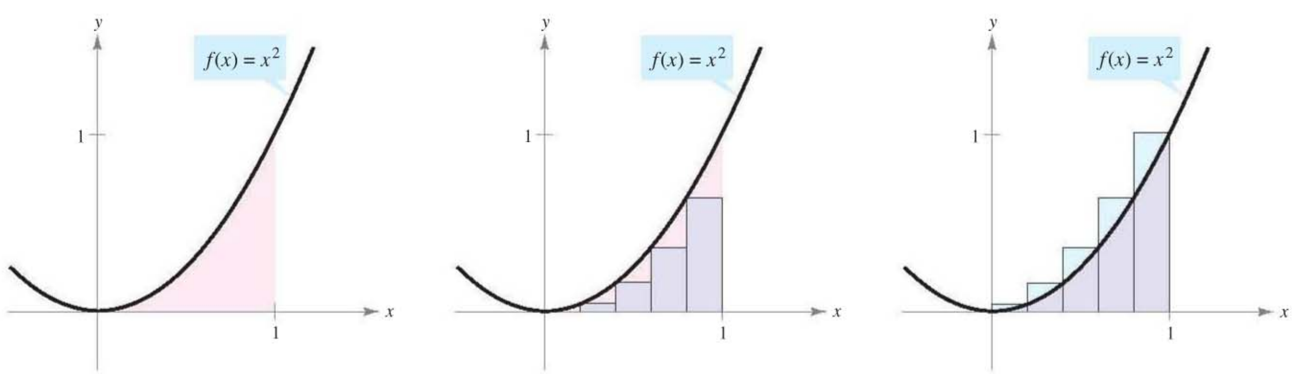



Approximation using four rectangles
Approximation using eight rectangles
**Figure 1.4**



**EXPLORATION**

Consider the region bounded by the graphs of $f(x) = x^2$, $y = 0$, and $x = 1$, as shown in part (a) of the figure. The area of the region can be approximated by two sets of rectangles—one set inscribed within the region and the other set circumscribed over the region, as shown in parts (b) and (c). Find the sum of the areas of each set of rectangles. Then use your results to approximate the area of the region.




**(a)** Bounded region
**(b)** Inscribed rectangles
**(c)** Circumscribed rectangles

## 1.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**In Exercises 1–5, decide whether the problem can be solved using precalculus or whether calculus is required. If the problem can be solved using precalculus, solve it. If the problem seems to require calculus, explain your reasoning and use a graphical or numerical approach to estimate the solution.**

**1.** Find the distance traveled in 15 seconds by an object traveling at a constant velocity of 20 feet per second.

**2.** Find the distance traveled in 15 seconds by an object moving with a velocity of $v(t) = 20 + 7 \cos t$ feet per second.

**3.** A bicyclist is riding on a path modeled by the function $f(x) = 0.04(8x - x^2)$, where $x$ and $f(x)$ are measured in miles. Find the rate of change of elevation at $x = 2$.

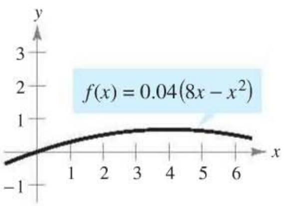



**Figure for 3**

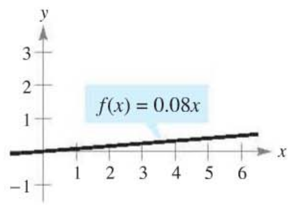
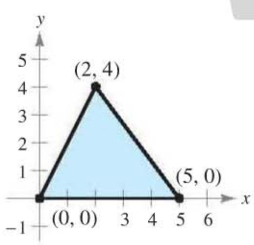



**Figure for 4**

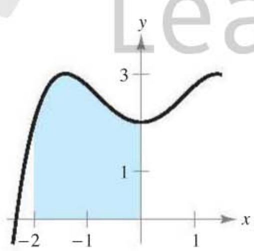
**4.** A bicyclist is riding on a path modeled by the function $f(x) = 0.08x$, where $x$ and $f(x)$ are measured in miles. Find the rate of change of elevation at $x = 2$.

**5.** Find the area of the shaded region.

(a)




(b)



**6.** ***Secant Lines*** Consider the function $f(x) = \sqrt{x}$ and the point $P(4, 2)$ on the graph of $f$.

(a) Graph $f$ and the secant lines passing through $P(4, 2)$ and $Q(x, f(x))$ for $x$-values of 1, 3, and 5.

(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of $f$ at $P(4, 2)$. Describe how to improve your approximation of the slope.

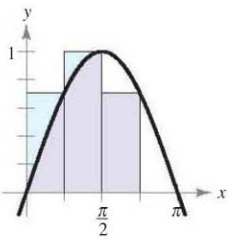
**7.** ***Secant Lines*** Consider the function $f(x) = 6x - x^2$ and the point $P(2, 8)$ on the graph of $f$.

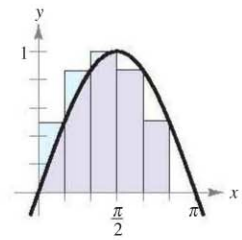
(a) Graph $f$ and the secant lines passing through $P(2, 8)$ and $Q(x, f(x))$ for $x$-values of 3, 2.5, and 1.5.

(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of $f$ at $P(2, 8)$. Describe how to improve your approximation of the slope.

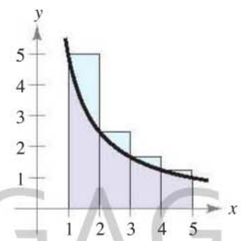
**8.** (a) Use the rectangles in each graph to approximate the area of the region bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.

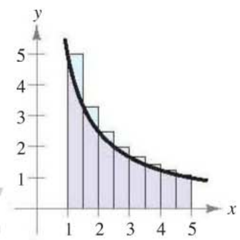




(b) Describe how you could continue this process to obtain a more accurate approximation of the area.

**9.** (a) Use the rectangles in each graph to approximate the area of the region bounded by $y = 5/x$, $y = 0$, $x = 1$, and $x = 5$.





(b) Describe how you could continue this process to obtain a more accurate approximation of the area.

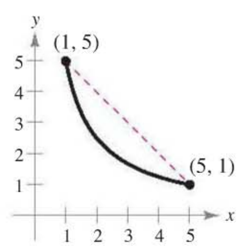


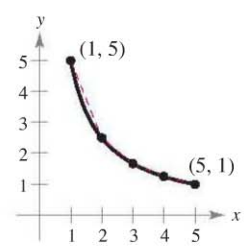
**CAPSTONE**

**10.** How would you describe the instantaneous rate of change of an automobile's position on the highway?

**WRITING ABOUT CONCEPTS**

**11.** Consider the length of the graph of $f(x) = 5/x$ from $(1, 5)$ to $(5, 1)$.




(a) Approximate the length of the curve by finding the distance between its two endpoints, as shown in the first figure.

(b) Approximate the length of the curve by finding the sum of the lengths of four line segments, as shown in the second figure.

(c) Describe how you could continue this process to obtain a more accurate approximation of the length of the curve.