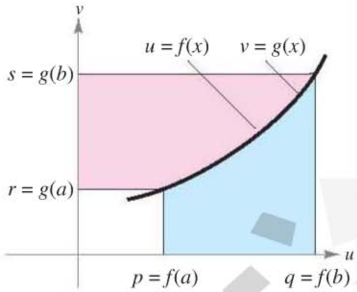
8.2 Integration by Parts

- Find an antiderivative using integration by parts.
- Use a tabular method to perform integration by parts.

EXPLORATION

Proof Without Words Here is a different approach to proving the formula for integration by parts. Exercise taken from "Proof Without Words: Integration by Parts" by Roger B. Nelsen, *Mathematics Magazine*, April 1991, by permission of the author.



Area (pink) + Area (blue) $= qs - pr$

$$\int_r^s u\,dv + \int_q^p v\,du = \Big[uv\Big]_{(p,r)}^{(q,s)}$$

$$\int_r^s u\,dv = \Big[uv\Big]_{(p,r)}^{(q,s)} - \int_q^p v\,du$$

Explain how this graph proves the theorem. Which notation in this proof is unfamiliar? What do you think it means?

Integration by Parts

In this section you will study an important integration technique called **integration by parts.** This technique can be applied to a wide variety of functions and is particularly useful for integrands involving *products* of algebraic and transcendental functions. For instance, integration by parts works well with integrals such as

$$\int x \ln x\,dx, \quad \int x^2 e^x\,dx, \quad \text{and} \quad \int e^x \sin x\,dx.$$

Integration by parts is based on the formula for the derivative of a product

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= uv' + vu'$$

where both u and v are differentiable functions of x. If u' and v' are continuous, you can integrate both sides of this equation to obtain

$$uv = \int uv'\,dx + \int vu'\,dx$$
$$= \int u\,dv + \int v\,du.$$

By rewriting this equation, you obtain the following theorem.

THEOREM 8.1 INTEGRATION BY PARTS

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du.$$

This formula expresses the original integral in terms of another integral. Depending on the choices of u and dv, it may be easier to evaluate the second integral than the original one. Because the choices of u and dv are critical in the integration by parts process, the following guidelines are provided.

GUIDELINES FOR INTEGRATION BY PARTS

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
2. Try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.

Note that dv always includes the dx of the original integrand.

EXAMPLE 1 Integration by Parts

Find $\int xe^x\,dx$.

Solution To apply integration by parts, you need to write the integral in the form $\int u\,dv$. There are several ways to do this.

$$\int \underbrace{(x)}_{u}\underbrace{(e^x dx)}_{dv}, \quad \int \underbrace{(e^x)}_{u}\underbrace{(x\,dx)}_{dv}, \quad \int \underbrace{(1)}_{u}\underbrace{(xe^x\,dx)}_{dv}, \quad \int \underbrace{(xe^x)}_{u}\underbrace{(dx)}_{dv}$$

The guidelines on page 527 suggest the first option because the derivative of $u = x$ is simpler than x, and $dv = e^x\,dx$ is the most complicated portion of the integrand that fits a basic integration formula.

$$dv = e^x\,dx \quad \Rightarrow \quad v = \int dv = \int e^x\,dx = e^x$$

$$u = x \quad \Rightarrow \quad du = dx$$

NOTE In Example 1, note that it is not necessary to include a constant of integration when solving

$$v = \int e^x\,dx = e^x + C_1.$$

To illustrate this, replace $v = e^x$ by $v = e^x + C_1$ and apply integration by parts to see that you obtain the same result.

Now, integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx \qquad \text{Substitute.}$$

$$= xe^x - e^x + C. \qquad \text{Integrate.}$$

To check this, differentiate $xe^x - e^x + C$ to see that you obtain the original integrand.

EXAMPLE 2 Integration by Parts

Find $\int x^2 \ln x\,dx$.

Solution In this case, x^2 is more easily integrated than $\ln x$. Furthermore, the derivative of $\ln x$ is simpler than $\ln x$. So, you should let $dv = x^2\,dx$.

$$dv = x^2\,dx \quad \Rightarrow \quad v = \int x^2\,dx = \frac{x^3}{3}$$

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}\,dx$$

Integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx \qquad \text{Substitute.}$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx \qquad \text{Simplify.}$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C. \qquad \text{Integrate.}$$

You can check this result by differentiating.

$$\frac{d}{dx}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right] = \frac{x^3}{3}\left(\frac{1}{x}\right) + (\ln x)(x^2) - \frac{x^2}{3} = x^2 \ln x$$

TECHNOLOGY Try graphing

$$\int x^2 \ln x\,dx \quad \text{and} \quad \frac{x^3}{3}\ln x - \frac{x^3}{9}$$

on your graphing utility. Do you get the same graph? (This will take a while, so be patient.)

FOR FURTHER INFORMATION To see how integration by parts is used to prove Stirling's approximation

$$\ln(n!) = n \ln n - n$$

see the article "The Validity of Stirling's Approximation: A Physical Chemistry Project" by A. S. Wallner and K. A. Brandt in *Journal of Chemical Education*.

One surprising application of integration by parts involves integrands consisting of single terms, such as $\int \ln x\, dx$ or $\int \arcsin x\, dx$. In these cases, try letting $dv = dx$, as shown in the next example.

EXAMPLE 3 An Integrand with a Single Term

Evaluate $\displaystyle\int_0^1 \arcsin x\, dx$.

Solution Let $dv = dx$.

$$dv = dx \quad \Rightarrow \quad v = \int dx = x$$

$$u = \arcsin x \quad \Rightarrow \quad du = \frac{1}{\sqrt{1-x^2}}\, dx$$

Integration by parts now produces

$$\int u\, dv = uv - \int v\, du \qquad \text{Integration by parts formula}$$

$$\int \arcsin x\, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\, dx \qquad \text{Substitute.}$$

$$= x \arcsin x + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\, dx \qquad \text{Rewrite.}$$

$$= x \arcsin x + \sqrt{1-x^2} + C. \qquad \text{Integrate.}$$

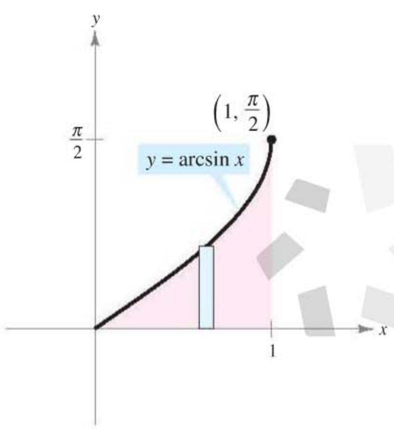
Using this antiderivative, you can evaluate the definite integral as follows.

$$\int_0^1 \arcsin x\, dx = \Big[x \arcsin x + \sqrt{1-x^2}\Big]_0^1$$

$$= \frac{\pi}{2} - 1$$

$$\approx 0.571$$

The area represented by this definite integral is shown in Figure 8.2. ■



The area of the region is approximately 0.571.
Figure 8.2

TECHNOLOGY Remember that there are two ways to use technology to evaluate a definite integral: (1) you can use a numerical approximation such as the Trapezoidal Rule or Simpson's Rule, or (2) you can use a computer algebra system to find the antiderivative and then apply the Fundamental Theorem of Calculus. Both methods have shortcomings. To find the possible error when using a numerical method, the integrand must have a second derivative (Trapezoidal Rule) or a fourth derivative (Simpson's Rule) in the interval of integration: the integrand in Example 3 fails to meet either of these requirements. To apply the Fundamental Theorem of Calculus, the symbolic integration utility must be able to find the antiderivative.

Which method would you use to evaluate

$$\int_0^1 \arctan x\, dx?$$

Which method would you use to evaluate

$$\int_0^1 \arctan x^2\, dx?$$

Some integrals require repeated use of the integration by parts formula.

EXAMPLE 4 Repeated Use of Integration by Parts

Find $\int x^2 \sin x\, dx$.

Solution The factors x^2 and $\sin x$ are equally easy to integrate. However, the derivative of x^2 becomes simpler, whereas the derivative of $\sin x$ does not. So, you should let $u = x^2$.

$$dv = \sin x\, dx \quad \Rightarrow \quad v = \int \sin x\, dx = -\cos x$$

$$u = x^2 \quad \Rightarrow \quad du = 2x\, dx$$

Now, integration by parts produces

$$\int x^2 \sin x\, dx = -x^2 \cos x + \int 2x \cos x\, dx. \qquad \text{First use of integration by parts}$$

This first use of integration by parts has succeeded in simplifying the original integral, but the integral on the right still doesn't fit a basic integration rule. To evaluate that integral, you can apply integration by parts again. This time, let $u = 2x$.

$$dv = \cos x\, dx \quad \Rightarrow \quad v = \int \cos x\, dx = \sin x$$

$$u = 2x \quad \Rightarrow \quad du = 2\, dx$$

Now, integration by parts produces

$$\begin{aligned}\int 2x \cos x\, dx &= 2x \sin x - \int 2 \sin x\, dx \qquad \text{Second use of integration by parts}\\ &= 2x \sin x + 2 \cos x + C.\end{aligned}$$

Combining these two results, you can write

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

■

When making repeated applications of integration by parts, you need to be careful not to interchange the substitutions in successive applications. For instance, in Example 4, the first substitution was $u = x^2$ and $dv = \sin x\, dx$. If, in the second application, you had switched the substitution to $u = \cos x$ and $dv = 2x$, you would have obtained

$$\begin{aligned}\int x^2 \sin x\, dx &= -x^2 \cos x + \int 2x \cos x\, dx\\ &= -x^2 \cos x + x^2 \cos x + \int x^2 \sin x\, dx = \int x^2 \sin x\, dx\end{aligned}$$

thereby undoing the previous integration and returning to the *original* integral. When making repeated applications of integration by parts, you should also watch for the appearance of a *constant multiple* of the original integral. For instance, this occurs when you use integration by parts to evaluate $\int e^x \cos 2x\, dx$, and also occurs in Example 5 on the next page.

The integral in Example 5 is an important one. In Section 8.4 (Example 5), you will see that it is used to find the arc length of a parabolic segment.

EXPLORATION

Try to find

$$\int e^x \cos 2x\, dx$$

by letting $u = \cos 2x$ and $dv = e^x\, dx$ in the first substitution. For the second substitution, let $u = \sin 2x$ and $dv = e^x\, dx$.

EXAMPLE 5 Integration by Parts

Find $\displaystyle\int \sec^3 x\,dx$.

Solution The most complicated portion of the integrand that can be easily integrated is $\sec^2 x$, so you should let $dv = \sec^2 x\,dx$ and $u = \sec x$.

$$dv = \sec^2 x\,dx \quad \Rightarrow \quad v = \int \sec^2 x\,dx = \tan x$$

$$u = \sec x \quad \Rightarrow \quad du = \sec x \tan x\,dx$$

Integration by parts produces

$$\int u\,dv = uv - \int v\,du \qquad \text{Integration by parts formula}$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx \qquad \text{Substitute.}$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx \qquad \text{Trigonometric identity}$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \qquad \text{Rewrite.}$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx \qquad \text{Collect like integrals.}$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C \qquad \text{Integrate.}$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C. \qquad \text{Divide by 2.}$$

STUDY TIP The trigonometric identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

play an important role in this chapter.

EXAMPLE 6 Finding a Centroid

A machine part is modeled by the region bounded by the graph of $y = \sin x$ and the x-axis, $0 \le x \le \pi/2$, as shown in Figure 8.3. Find the centroid of this region.

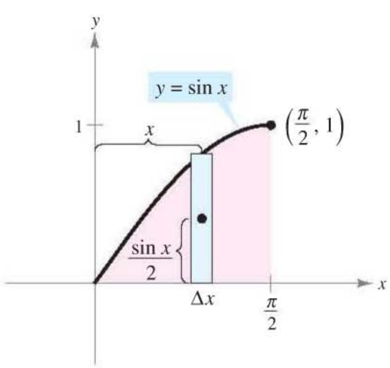


Figure 8.3

Solution Begin by finding the area of the region.

$$A = \int_0^{\pi/2} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2} = 1$$

Now, you can find the coordinates of the centroid as follows.

$$\bar{y} = \frac{1}{A}\int_0^{\pi/2} \frac{\sin x}{2}(\sin x)\,dx = \frac{1}{4}\int_0^{\pi/2} (1 - \cos 2x)\,dx = \frac{1}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{8}$$

You can evaluate the integral for $\bar{x}$, $(1/A)\int_0^{\pi/2} x \sin x\,dx$, with integration by parts. To do this, let $dv = \sin x\,dx$ and $u = x$. This produces $v = -\cos x$ and $du = dx$, and you can write

$$\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

Finally, you can determine $\bar{x}$ to be

$$\bar{x} = \frac{1}{A}\int_0^{\pi/2} x \sin x\,dx = \Big[-x\cos x + \sin x\Big]_0^{\pi/2} = 1.$$

So, the centroid of the region is $(1, \pi/8)$. ■

As you gain experience in using integration by parts, your skill in determining u and dv will increase. The following summary lists several common integrals with suggestions for the choices of u and dv.

STUDY TIP You can use the acronym LIATE as a guideline for choosing u in integration by parts. In order, check the integrand for the following.

Is there a Logarithmic part?

Is there an Inverse trigonometric part?

Is there an Algebraic part?

Is there a Trigonometric part?

Is there an Exponential part?

SUMMARY OF COMMON INTEGRALS USING INTEGRATION BY PARTS

1. For integrals of the form

$$\int x^n e^{ax}\, dx, \qquad \int x^n \sin ax\, dx, \qquad \text{or} \qquad \int x^n \cos ax\, dx$$

let $u = x^n$ and let $dv = e^{ax}\, dx$, $\sin ax\, dx$, or $\cos ax\, dx$.

2. For integrals of the form

$$\int x^n \ln x\, dx, \qquad \int x^n \arcsin ax\, dx, \qquad \text{or} \qquad \int x^n \arctan ax\, dx$$

let $u = \ln x$, $\arcsin ax$, or $\arctan ax$ and let $dv = x^n\, dx$.

3. For integrals of the form

$$\int e^{ax} \sin bx\, dx \qquad \text{or} \qquad \int e^{ax} \cos bx\, dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax}\, dx$.

Tabular Method

In problems involving repeated applications of integration by parts, a tabular method, illustrated in Example 7, can help to organize the work. This method works well for integrals of the form $\int x^n \sin ax\, dx$, $\int x^n \cos ax\, dx$, and $\int x^n e^{ax}\, dx$.

EXAMPLE 7 Using the Tabular Method

Find $\displaystyle\int x^2 \sin 4x\, dx$.

Solution Begin as usual by letting $u = x^2$ and $dv = v'\, dx = \sin 4x\, dx$. Next, create a table consisting of three columns, as shown.

Alternate Signs	*u and Its Derivatives*	*v′ and Its Antiderivatives*
$+$	x^2	$\sin 4x$
$-$	$2x$	$-\frac{1}{4}\cos 4x$
$+$	2	$-\frac{1}{16}\sin 4x$
$-$	0	$\frac{1}{64}\cos 4x$

Differentiate until you obtain 0 as a derivative.

The solution is obtained by adding the signed products of the diagonal entries:

$$\int x^2 \sin 4x\, dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x + C.$$

■ **FOR FURTHER INFORMATION** For more information on the tabular method, see the article "Tabular Integration by Parts" by David Horowitz in *The College Mathematics Journal*, and the article "More on Tabular Integration by Parts" by Leonard Gillman in *The College Mathematics Journal*. To view these articles, go to the website *www.matharticles.com*.

8.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, identify u and dv for finding the integral using integration by parts. (Do not evaluate the integral.)

1. $\int xe^{2x}\,dx$

2. $\int x^2 e^{2x}\,dx$

3. $\int (\ln x)^2\,dx$

4. $\int \ln 5x\,dx$

5. $\int x\sec^2 x\,dx$

6. $\int x^2\cos x\,dx$

In Exercises 7–10, evaluate the integral using integration by parts with the given choices of u and dv.

7. $\displaystyle\int x^3 \ln x\,dx;\ u = \ln x,\ dv = x^3\,dx$

8. $\displaystyle\int (4x + 7)e^x\,dx;\ u = 4x + 7,\ dv = e^x\,dx$

9. $\displaystyle\int x\sin 3x\,dx;\ u = x,\ dv = \sin 3x\,dx$

10. $\displaystyle\int x\cos 4x\,dx;\ u = x,\ dv = \cos 4x\,dx$

In Exercises 11–38, find the integral. (*Note:* Solve by the simplest method—not all require integration by parts.)

11. $\displaystyle\int xe^{-4x}\,dx$

12. $\displaystyle\int \frac{4x}{e^x}\,dx$

13. $\displaystyle\int x^3e^x\,dx$

14. $\displaystyle\int \frac{e^{1/t}}{t^2}\,dt$

15. $\displaystyle\int x^2e^{x^3}\,dx$

16. $\displaystyle\int x^4\ln x\,dx$

17. $\displaystyle\int t\ln(t + 1)\,dt$

18. $\displaystyle\int \frac{1}{x(\ln x)^3}\,dx$

19. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

20. $\displaystyle\int \frac{\ln x}{x^2}\,dx$

21. $\displaystyle\int \frac{xe^{2x}}{(2x + 1)^2}\,dx$

22. $\displaystyle\int \frac{x^3 e^{x^2}}{(x^2 + 1)^2}\,dx$

23. $\displaystyle\int (x^2 - 1)e^x\,dx$

24. $\displaystyle\int \frac{\ln 2x}{x^2}\,dx$

25. $\displaystyle\int x\sqrt{x - 5}\,dx$

26. $\displaystyle\int \frac{x}{\sqrt{5 + 4x}}\,dx$

27. $\displaystyle\int x\cos x\,dx$

28. $\displaystyle\int x\sin x\,dx$

29. $\displaystyle\int x^3\sin x\,dx$

30. $\displaystyle\int x^2\cos x\,dx$

31. $\displaystyle\int t\csc t\cot t\,dt$

32. $\displaystyle\int \theta\sec\theta\tan\theta\,d\theta$

33. $\displaystyle\int \arctan x\,dx$

34. $\displaystyle\int 4\arccos x\,dx$

35. $\displaystyle\int e^{2x}\sin x\,dx$

36. $\displaystyle\int e^{-3x}\sin 5x\,dx$

37. $\displaystyle\int e^{-x}\cos 2x\,dx$

38. $\displaystyle\int e^{3x}\cos 4x\,dx$

In Exercises 39–44, solve the differential equation.

39. $y' = xe^{x^2}$

40. $y' = \ln x$

41. $\displaystyle\frac{dy}{dt} = \frac{t^2}{\sqrt{3 + 5t}}$

42. $\displaystyle\frac{dy}{dx} = x^2\sqrt{x - 3}$

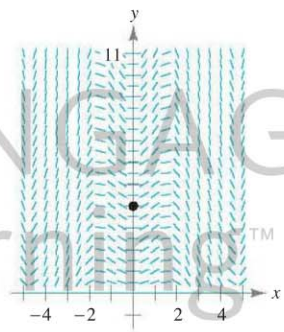
43. $(\cos y)y' = 2x$

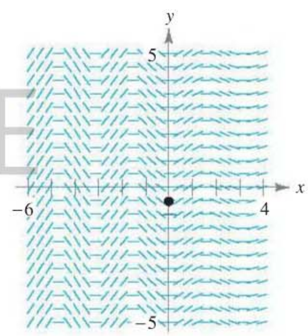
44. $\displaystyle y' = \arctan\frac{x}{2}$

Slope Fields **In Exercises 45 and 46, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

45. $\displaystyle\frac{dy}{dx} = x\sqrt{y}\cos x,\ (0, 4)$

46. $\displaystyle\frac{dy}{dx} = e^{-x/3}\sin 2x,\ \left(0, -\tfrac{18}{37}\right)$





CAS ***Slope Fields*** **In Exercises 47 and 48, use a computer algebra system to graph the slope field for the differential equation and graph the solution through the specified initial condition.**

47. $\displaystyle\frac{dy}{dx} = \frac{x}{y}e^{x/8}$

$y(0) = 2$

48. $\displaystyle\frac{dy}{dx} = \frac{x}{y}\sin x$

$y(0) = 4$

In Exercises 49–60, evaluate the definite integral. Use a graphing utility to confirm your result.

49. $\displaystyle\int_0^3 xe^{x/2}\,dx$

50. $\displaystyle\int_0^2 x^2 e^{-2x}\,dx$

51. $\displaystyle\int_0^{\pi/4} x\cos 2x\,dx$

52. $\displaystyle\int_0^{\pi} x\sin 2x\,dx$

53. $\displaystyle\int_0^{1/2} \arccos x\,dx$

54. $\displaystyle\int_0^1 x\arcsin x^2\,dx$

55. $\displaystyle\int_0^1 e^x\sin x\,dx$

56. $\displaystyle\int_0^2 e^{-x}\cos x\,dx$

57. $\displaystyle\int_1^2 \sqrt{x}\ln x\,dx$

58. $\displaystyle\int_0^1 \ln(4 + x^2)\,dx$

59. $\displaystyle\int_2^4 x\,\mathrm{arcsec}\,x\,dx$

60. $\displaystyle\int_0^{\pi/8} x\sec^2 2x\,dx$

In Exercises 61–66, use the tabular method to find the integral.

61. $\int x^2e^{2x}\,dx$

62. $\int x^3e^{-2x}\,dx$

63. $\int x^3 \sin x\,dx$

64. $\int x^3 \cos 2x\,dx$

65. $\int x \sec^2 x\,dx$

66. $\int x^2(x-2)^{3/2}\,dx$

In Exercises 67–74, find or evaluate the integral using substitution first, then using integration by parts.

67. $\int \sin\sqrt{x}\,dx$

68. $\int \cos\sqrt{x}\,dx$

69. $\int_0^4 x\sqrt{4-x}\,dx$

70. $\int 2x^3 \cos x^2\,dx$

71. $\int x^5e^{x^2}\,dx$

72. $\int_0^2 e^{\sqrt{2x}}\,dx$

73. $\int \cos(\ln x)\,dx$

74. $\int \ln(x^2+1)\,dx$

WRITING ABOUT CONCEPTS

75. Integration by parts is based on what differentiation rule? Explain.

76. In your own words, state how you determine which parts of the integrand should be u and dv.

77. When evaluating $\int x \sin x\,dx$, explain how letting $u = \sin x$ and $dv = x\,dx$ makes the solution more difficult to find.

CAPSTONE

78. State whether you would use integration by parts to evaluate each integral. If so, identify what you would use for u and dv. Explain your reasoning.

(a) $\int \frac{\ln x}{x}\,dx$ (b) $\int x \ln x\,dx$ (c) $\int x^2e^{-3x}\,dx$

(d) $\int 2x\,e^{x^2}\,dx$ (e) $\int \frac{x}{\sqrt{x+1}}\,dx$ (f) $\int \frac{x}{\sqrt{x^2+1}}\,dx$

CAS **In Exercises 79–82, use a computer algebra system to (a) find or evaluate the integral and (b) graph two antiderivatives. (c) Describe the relationship between the graphs of the antiderivatives.**

79. $\int t^3\,e^{-4t}\,dt$

80. $\int \alpha^4 \sin \pi\alpha\,d\alpha$

81. $\int_0^{\pi/2} e^{-2x} \sin 3x\,dx$

82. $\int_0^5 x^4(25-x^2)^{3/2}\,dx$

83. Integrate $\int 2x\sqrt{2x-3}\,dx$

(a) by parts, letting $dv = \sqrt{2x-3}\,dx$.

(b) by substitution, letting $u = 2x-3$.

84. Integrate $\int x\sqrt{9+x}\,dx$

(a) by parts, letting $dv = \sqrt{9+x}\,dx$.

(b) by substitution, letting $u = 9+x$.

85. Integrate $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

(a) by parts, letting $dv = \left(x/\sqrt{4+x^2}\right)dx$.

(b) by substitution, letting $u = 4+x^2$.

86. Integrate $\int x\sqrt{4-x}\,dx$

(a) by parts, letting $dv = \sqrt{4-x}\,dx$.

(b) by substitution, letting $u = 4-x$.

CAS **In Exercises 87 and 88, use a computer algebra system to find the integrals for $n = 0, 1, 2,$ and 3. Use the result to obtain a general rule for the integrals for any positive integer n and test your results for $n = 4$.**

87. $\int x^n \ln x\,dx$

88. $\int x^ne^x\,dx$

In Exercises 89–94, use integration by parts to prove the formula. (For Exercises 89–92, assume that n is a positive integer.)

89. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1} \cos x\,dx$

90. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1} \sin x\,dx$

91. $\int x^n \ln x\,dx = \frac{x^{n+1}}{(n+1)^2}[-1 + (n+1)\ln x] + C$

92. $\int x^n\,e^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}\,e^{ax}\,dx$

93. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} + C$

94. $\int e^{ax} \cos bx\,dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2} + C$

In Exercises 95–98, find the integral by using the appropriate formula from Exercises 89–94.

95. $\int x^5 \ln x\,dx$

96. $\int x^2 \cos x\,dx$

97. $\int e^{2x} \cos 3x\,dx$

98. $\int x^3e^{2x}\,dx$

Area **In Exercises 99–102, use a graphing utility to graph the region bounded by the graphs of the equations, and find the area of the region.**

99. $y = 2xe^{-x}, y = 0, x = 3$

100. $y = \frac{1}{16}xe^{-x/4}, y = 0, x = 0, x = 4$

101. $y = e^{-x} \sin \pi x, y = 0, x = 1$

102. $y = x \sin x, y = 0, x = \pi$

103. ***Area, Volume, and Centroid*** Given the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$, find

(a) the area of the region.

(b) the volume of the solid generated by revolving the region about the x-axis.

(c) the volume of the solid generated by revolving the region about the y-axis.

(d) the centroid of the region.

104. ***Volume and Centroid*** Given the region bounded by the graphs of $y = x \sin x$, $y = 0$, $x = 0$, and $x = \pi$, find

(a) the volume of the solid generated by revolving the region about the x-axis.

(b) the volume of the solid generated by revolving the region about the y-axis.

(c) the centroid of the region.

105. ***Centroid*** Find the centroid of the region bounded by the graphs of $y = \arcsin x$, $x = 0$, and $y = \pi/2$. How is this problem related to Example 6 in this section?

106. ***Centroid*** Find the centroid of the region bounded by the graphs of $f(x) = x^2$, $g(x) = 2^x$, $x = 2$, and $x = 4$.

107. ***Average Displacement*** A damping force affects the vibration of a spring so that the displacement of the spring is given by $y = e^{-4t}(\cos 2t + 5 \sin 2t)$. Find the average value of y on the interval from $t = 0$ to $t = \pi$.

108. ***Memory Model*** A model for the ability M of a child to memorize, measured on a scale from 0 to 10, is given by $M = 1 + 1.6t \ln t$, $0 < t \le 4$, where t is the child's age in years. Find the average value of this model

(a) between the child's first and second birthdays.

(b) between the child's third and fourth birthdays.

Present Value **In Exercises 109 and 110, find the present value P of a continuous income flow of $c(t)$ dollars per year if**

$$P = \int_0^{t_1} c(t)e^{-rt}\,dt$$

where t_1 is the time in years and r is the annual interest rate compounded continuously.

109. $c(t) = 100{,}000 + 4000t$, $r = 5\%$, $t_1 = 10$

110. $c(t) = 30{,}000 + 500t$, $r = 7\%$, $t_1 = 5$

Integrals Used to Find Fourier Coefficients **In Exercises 111 and 112, verify the value of the definite integral, where n is a positive integer.**

111. $$\int_{-\pi}^{\pi} x \sin nx\,dx = \begin{cases} \dfrac{2\pi}{n}, & n \text{ is odd} \\ -\dfrac{2\pi}{n}, & n \text{ is even} \end{cases}$$

112. $$\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{(-1)^n 4\pi}{n^2}$$

113. ***Vibrating String*** A string stretched between the two points $(0, 0)$ and $(2, 0)$ is plucked by displacing the string h units at its midpoint. The motion of the string is modeled by a **Fourier Sine Series** whose coefficients are given by

$$b_n = h\int_0^1 x \sin\frac{n\pi x}{2}\,dx + h\int_1^2 (-x + 2)\sin\frac{n\pi x}{2}\,dx.$$

Find b_n.

114. Find the fallacy in the following argument that $0 = 1$.

$$dv = dx \quad \Rightarrow \quad v = \int dx = x$$

$$u = \frac{1}{x} \quad \Rightarrow \quad du = -\frac{1}{x^2}\,dx$$

$$0 + \int \frac{dx}{x} = \left(\frac{1}{x}\right)(x) - \int\left(-\frac{1}{x^2}\right)(x)\,dx = 1 + \int \frac{dx}{x}$$

So, $0 = 1$.

115. Let $y = f(x)$ be positive and strictly increasing on the interval $0 < a \le x \le b$. Consider the region R bounded by the graphs of $y = f(x)$, $y = 0$, $x = a$, and $x = b$. If R is revolved about the y-axis, show that the disk method and shell method yield the same volume.

116. ***Euler's Method*** Consider the differential equation $f'(x) = xe^{-x}$ with the initial condition $f(0) = 0$.

(a) Use integration to solve the differential equation.

(b) Use a graphing utility to graph the solution of the differential equation.

(c) Use Euler's Method with $h = 0.05$, and the recursive capabilities of a graphing utility, to generate the first 80 points of the graph of the approximate solution. Use the graphing utility to plot the points. Compare the result with the graph in part (b).

(d) Repeat part (c) using $h = 0.1$ and generate the first 40 points.

(e) Why is the result in part (c) a better approximation of the solution than the result in part (d)?

Euler's Method **In Exercises 117 and 118, consider the differential equation and repeat parts (a)–(d) of Exercise 116.**

117. $f'(x) = 3x \sin(2x)$, $f(0) = 0$

118. $f'(x) = \cos\sqrt{x}$, $f(0) = 1$

119. ***Think About It*** Give a geometric explanation of why

$$\int_0^{\pi/2} x \sin x\,dx \le \int_0^{\pi/2} x\,dx.$$

Verify the inequality by evaluating the integrals.

120. ***Finding a Pattern*** Find the area bounded by the graphs of $y = x \sin x$ and $y = 0$ over each interval.

(a) $[0, \pi]$ (b) $[\pi, 2\pi]$ (c) $[2\pi, 3\pi]$

Describe any patterns that you notice. What is the area between the graphs of $y = x \sin x$ and $y = 0$ over the interval $[n\pi, (n + 1)\pi]$, where n is any nonnegative integer? Explain.