8.1 Basic Integration Rules

■ **Review procedures for fitting an integrand to one of the basic integration rules.**

Fitting Integrands to Basic Integration Rules

In this chapter, you will study several integration techniques that greatly expand the set of integrals to which the basic integration rules can be applied. These rules are reviewed on page 522. A major step in solving any integration problem is recognizing which basic integration rule to use. As shown in Example 1, slight differences in the integrand can lead to very different solution techniques.

EXPLORATION

A Comparison of Three Similar Integrals Which, if any, of the following integrals can be evaluated using the 20 basic integration rules? For any that can be evaluated, do so. For any that can't, explain why.

a. $\displaystyle\int \frac{3}{\sqrt{1-x^2}}\,dx$

b. $\displaystyle\int \frac{3x}{\sqrt{1-x^2}}\,dx$

c. $\displaystyle\int \frac{3x^2}{\sqrt{1-x^2}}\,dx$

EXAMPLE 1 A Comparison of Three Similar Integrals

Find each integral.

a. $\displaystyle\int \frac{4}{x^2+9}\,dx$ **b.** $\displaystyle\int \frac{4x}{x^2+9}\,dx$ **c.** $\displaystyle\int \frac{4x^2}{x^2+9}\,dx$

Solution

a. Use the Arctangent Rule and let $u = x$ and $a = 3$.

$$\int \frac{4}{x^2+9}\,dx = 4\int \frac{1}{x^2+3^2}\,dx \qquad \text{Constant Multiple Rule}$$

$$= 4\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C \qquad \text{Arctangent Rule}$$

$$= \frac{4}{3}\arctan\frac{x}{3} + C \qquad \text{Simplify.}$$

b. Here the Arctangent Rule does not apply because the numerator contains a factor of x. Consider the Log Rule and let $u = x^2 + 9$. Then $du = 2x\,dx$, and you have

$$\int \frac{4x}{x^2+9}\,dx = 2\int \frac{2x\,dx}{x^2+9} \qquad \text{Constant Multiple Rule}$$

$$= 2\int \frac{du}{u} \qquad \text{Substitution: } u = x^2+9$$

$$= 2\ln|u| + C = 2\ln(x^2+9) + C. \qquad \text{Log Rule}$$

c. Because the degree of the numerator is equal to the degree of the denominator, you should first use division to rewrite the improper rational function as the sum of a polynomial and a proper rational function.

$$\int \frac{4x^2}{x^2+9}\,dx = \int \left(4 - \frac{36}{x^2+9}\right) dx \qquad \text{Rewrite using long division.}$$

$$= \int 4\,dx - 36\int \frac{1}{x^2+9}\,dx \qquad \text{Write as two integrals.}$$

$$= 4x - 36\left(\frac{1}{3}\arctan\frac{x}{3}\right) + C \qquad \text{Integrate.}$$

$$= 4x - 12\arctan\frac{x}{3} + C \qquad \text{Simplify.}$$ ■

NOTE Notice in Example 1(c) that some preliminary algebra is required before applying the rules for integration, and that subsequently more than one rule is needed to evaluate the resulting integral.

The icon indicates that you will find a CAS Investigation on the book's website and the online Eduspace® *system. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems* Maple *and* Mathematica.

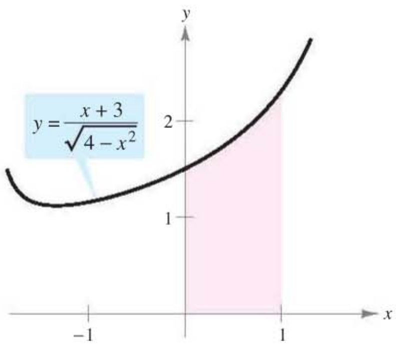
EXAMPLE 2 Using Two Basic Rules to Solve a Single Integral

Evaluate $\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx$.

Solution Begin by writing the integral as the sum of two integrals. Then apply the Power Rule and the Arcsine Rule, as follows.

$$\begin{aligned}\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx &= \int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^1 \frac{3}{\sqrt{4-x^2}}\,dx \\ &= -\frac{1}{2}\int_0^1 (4-x^2)^{-1/2}(-2x)\,dx + 3\int_0^1 \frac{1}{\sqrt{2^2-x^2}}\,dx \\ &= \left[-(4-x^2)^{1/2} + 3\arcsin\frac{x}{2}\right]_0^1 \\ &= \left(-\sqrt{3}+\frac{\pi}{2}\right) - (-2+0) \\ &\approx 1.839\end{aligned}$$

See Figure 8.1. ■



The area of the region is approximately 1.839.
Figure 8.1

TECHNOLOGY Simpson's Rule can be used to give a good approximation of the value of the integral in Example 2 (for $n = 10$, the approximation is 1.839). When using numerical integration, however, you should be aware that Simpson's Rule does not always give good approximations when one or both of the limits of integration are near a vertical asymptote. For instance, using the Fundamental Theorem of Calculus, you can obtain

$$\int_0^{1.99} \frac{x+3}{\sqrt{4-x^2}}\,dx \approx 6.213.$$

Applying Simpson's Rule (with $n = 10$) to this integral produces an approximation of 6.889.

EXAMPLE 3 A Substitution Involving $a^2 - u^2$

Find $\int \frac{x^2}{\sqrt{16-x^6}}\,dx$.

Solution Because the radical in the denominator can be written in the form

$$\sqrt{a^2-u^2} = \sqrt{4^2-(x^3)^2}$$

you can try the substitution $u = x^3$. Then $du = 3x^2\,dx$, and you have

$$\begin{aligned}\int \frac{x^2}{\sqrt{16-x^6}}\,dx &= \frac{1}{3}\int \frac{3x^2\,dx}{\sqrt{16-(x^3)^2}} && \text{Rewrite integral.} \\ &= \frac{1}{3}\int \frac{du}{\sqrt{4^2-u^2}} && \text{Substitution: } u = x^3 \\ &= \frac{1}{3}\arcsin\frac{u}{4} + C && \text{Arcsine Rule} \\ &= \frac{1}{3}\arcsin\frac{x^3}{4} + C. && \text{Rewrite as a function of } x.\end{aligned}$$
■

STUDY TIP Rules 18, 19, and 20 of the basic integration rules on the next page all have expressions involving the sum or difference of two squares:

$a^2 - u^2$

$a^2 + u^2$

$u^2 - a^2$

These expressions are often apparent after a u-substitution, as shown in Example 3.

Surprisingly, two of the most commonly overlooked integration rules are the Log Rule and the Power Rule. Notice in the next two examples how these two integration rules can be disguised.

REVIEW OF BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

EXAMPLE 4 A Disguised Form of the Log Rule

Find $\int \frac{1}{1 + e^x}\,dx$.

Solution The integral does not appear to fit any of the basic rules. However, the quotient form suggests the Log Rule. If you let $u = 1 + e^x$, then $du = e^x\,dx$. You can obtain the required du by adding and subtracting e^x in the numerator, as follows.

$$\int \frac{1}{1 + e^x}\,dx = \int \frac{1 + e^x - e^x}{1 + e^x}\,dx \qquad \text{Add and subtract } e^x \text{ in numerator.}$$

$$= \int \left(\frac{1 + e^x}{1 + e^x} - \frac{e^x}{1 + e^x}\right)dx \qquad \text{Rewrite as two fractions.}$$

$$= \int dx - \int \frac{e^x\,dx}{1 + e^x} \qquad \text{Rewrite as two integrals.}$$

$$= x - \ln(1 + e^x) + C \qquad \text{Integrate.}$$

■

NOTE There is usually more than one way to solve an integration problem. For instance, in Example 4, try integrating by multiplying the numerator and denominator by e^{-x} to obtain an integral of the form $-\int du/u$. See if you can get the same answer by this procedure. (Be careful: the answer will appear in a different form.) ■

EXAMPLE 5 A Disguised Form of the Power Rule

Find $\int (\cot x)[\ln(\sin x)]\,dx$.

Solution Again, the integral does not appear to fit any of the basic rules. However, considering the two primary choices for u $[u = \cot x$ and $u = \ln(\sin x)]$, you can see that the second choice is the appropriate one because

$$u = \ln(\sin x) \quad \text{and} \quad du = \frac{\cos x}{\sin x}\,dx = \cot x\,dx.$$

So,

$$\int (\cot x)[\ln(\sin x)]\,dx = \int u\,du \qquad \text{Substitution: } u = \ln(\sin x)$$

$$= \frac{u^2}{2} + C \qquad \text{Integrate.}$$

$$= \frac{1}{2}[\ln(\sin x)]^2 + C. \qquad \text{Rewrite as a function of } x.$$

■

NOTE In Example 5, try *checking* that the derivative of

$$\frac{1}{2}[\ln(\sin x)]^2 + C$$

is the integrand of the original integral. ■

Trigonometric identities can often be used to fit integrals to one of the basic integration rules.

EXAMPLE 6 Using Trigonometric Identities

Find $\int \tan^2 2x\, dx$.

Solution Note that $\tan^2 u$ is not in the list of basic integration rules. However, $\sec^2 u$ is in the list. This suggests the trigonometric identity $\tan^2 u = \sec^2 u - 1$. If you let $u = 2x$, then $du = 2\, dx$ and

$$\int \tan^2 2x\, dx = \frac{1}{2}\int \tan^2 u\, du \qquad \text{Substitution: } u = 2x$$

$$= \frac{1}{2}\int (\sec^2 u - 1)\, du \qquad \text{Trigonometric identity}$$

$$= \frac{1}{2}\int \sec^2 u\, du - \frac{1}{2}\int du \qquad \text{Rewrite as two integrals.}$$

$$= \frac{1}{2}\tan u - \frac{u}{2} + C \qquad \text{Integrate.}$$

$$= \frac{1}{2}\tan 2x - x + C. \qquad \text{Rewrite as a function of } x.$$

TECHNOLOGY If you have access to a computer algebra system, try using it to evaluate the integrals in this section. Compare the *forms* of the antiderivatives given by the software with the forms obtained by hand. Sometimes the forms will be the same, but often they will differ. For instance, why is the antiderivative $\ln 2x + C$ equivalent to the antiderivative $\ln x + C$?

This section concludes with a summary of the common procedures for fitting integrands to the basic integration rules.

PROCEDURES FOR FITTING INTEGRANDS TO BASIC INTEGRATION RULES

Technique	*Example*
Expand (numerator).	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\dfrac{1 + x}{x^2 + 1} = \dfrac{1}{x^2 + 1} + \dfrac{x}{x^2 + 1}$
Complete the square.	$\dfrac{1}{\sqrt{2x - x^2}} = \dfrac{1}{\sqrt{1 - (x - 1)^2}}$
Divide improper rational function.	$\dfrac{x^2}{x^2 + 1} = 1 - \dfrac{1}{x^2 + 1}$
Add and subtract terms in numerator.	$\dfrac{2x}{x^2 + 2x + 1} = \dfrac{2x + 2 - 2}{x^2 + 2x + 1} = \dfrac{2x + 2}{x^2 + 2x + 1} - \dfrac{2}{(x + 1)^2}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate.	$\dfrac{1}{1 + \sin x} = \left(\dfrac{1}{1 + \sin x}\right)\left(\dfrac{1 - \sin x}{1 - \sin x}\right) = \dfrac{1 - \sin x}{1 - \sin^2 x}$ $= \dfrac{1 - \sin x}{\cos^2 x} = \sec^2 x - \dfrac{\sin x}{\cos^2 x}$

NOTE Remember that you can separate numerators but not denominators. Watch out for this common error when fitting integrands to basic rules.

$$\frac{1}{x^2 + 1} \neq \frac{1}{x^2} + \frac{1}{1} \qquad \text{Do not separate denominators.}$$

8.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, select the correct antiderivative.

1. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+1}}$
   (a) $2\sqrt{x^2+1} + C$ (b) $\sqrt{x^2+1} + C$
   (c) $\frac{1}{2}\sqrt{x^2+1} + C$ (d) $\ln(x^2+1) + C$

2. $\dfrac{dy}{dx} = \dfrac{x}{x^2+1}$
   (a) $\ln\sqrt{x^2+1} + C$ (b) $\dfrac{2x}{(x^2+1)^2} + C$
   (c) $\arctan x + C$ (d) $\ln(x^2+1) + C$

3. $\dfrac{dy}{dx} = \dfrac{1}{x^2+1}$
   (a) $\ln\sqrt{x^2+1} + C$ (b) $\dfrac{2x}{(x^2+1)^2} + C$
   (c) $\arctan x + C$ (d) $\ln(x^2+1) + C$

4. $\dfrac{dy}{dx} = x\cos(x^2+1)$
   (a) $2x\sin(x^2+1) + C$ (b) $-\frac{1}{2}\sin(x^2+1) + C$
   (c) $\frac{1}{2}\sin(x^2+1) + C$ (d) $-2x\sin(x^2+1) + C$

In Exercises 5–14, select the basic integration formula you can use to find the integral, and identify *u* and *a* when appropriate.

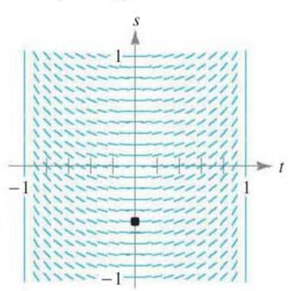
5. $\int (5x-3)^4\,dx$
6. $\int \dfrac{2t+1}{t^2+t-4}\,dt$
7. $\int \dfrac{1}{\sqrt{x}(1-2\sqrt{x})}\,dx$
8. $\int \dfrac{2}{(2t-1)^2+4}\,dt$
9. $\int \dfrac{3}{\sqrt{1-t^2}}\,dt$
10. $\int \dfrac{-2x}{\sqrt{x^2-4}}\,dx$
11. $\int t\sin t^2\,dt$
12. $\int \sec 5x\tan 5x\,dx$
13. $\int (\cos x)e^{\sin x}\,dx$
14. $\int \dfrac{1}{x\sqrt{x^2-4}}\,dx$

In Exercises 15–52, find the indefinite integral.

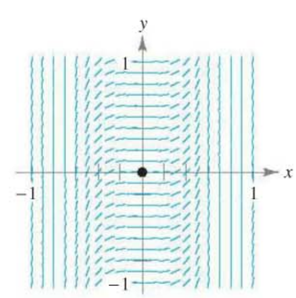
15. $\int 14(x-5)^6\,dx$
16. $\int \dfrac{9}{(t-8)^2}\,dt$
17. $\int \dfrac{7}{(z-10)^7}\,dz$
18. $\int t^2\sqrt[3]{t^3-1}\,dt$
19. $\int \left[v + \dfrac{1}{(3v-1)^3}\right]dv$
20. $\int \left[x - \dfrac{5}{(3x+5)^2}\right]dx$
21. $\int \dfrac{t^2-3}{-t^3+9t+1}\,dt$
22. $\int \dfrac{x+1}{\sqrt{x^2+2x-4}}\,dx$
23. $\int \dfrac{x^2}{x-1}\,dx$
24. $\int \dfrac{4x}{x-8}\,dx$
25. $\int \dfrac{e^x}{1+e^x}\,dx$
26. $\int \left(\dfrac{1}{7x-2} - \dfrac{1}{7x+2}\right)dx$
27. $\int (5+4x^2)^2\,dx$
28. $\int x\left(1+\dfrac{1}{x}\right)^3 dx$
29. $\int x\cos 2\pi x^2\,dx$
30. $\int \sec 4x\,dx$
31. $\int \csc \pi x\cot \pi x\,dx$
32. $\int \dfrac{\sin x}{\sqrt{\cos x}}\,dx$
33. $\int e^{11x}\,dx$
34. $\int \csc^2 x e^{\cot x}\,dx$
35. $\int \dfrac{2}{e^{-x}+1}\,dx$
36. $\int \dfrac{5}{3e^x-2}\,dx$
37. $\int \dfrac{\ln x^2}{x}\,dx$
38. $\int (\tan x)[\ln(\cos x)]\,dx$
39. $\int \dfrac{1+\sin x}{\cos x}\,dx$
40. $\int \dfrac{1+\cos\alpha}{\sin\alpha}\,d\alpha$
41. $\int \dfrac{1}{\cos\theta - 1}\,d\theta$
42. $\int \dfrac{2}{3(\sec x - 1)}\,dx$
43. $\int \dfrac{-1}{\sqrt{1-(4t+1)^2}}\,dt$
44. $\int \dfrac{1}{9+5x^2}\,dx$
45. $\int \dfrac{\tan(2/t)}{t^2}\,dt$
46. $\int \dfrac{e^{1/t}}{t^2}\,dt$
47. $\int \dfrac{6}{\sqrt{10x-x^2}}\,dx$
48. $\int \dfrac{1}{(x-1)\sqrt{4x^2-8x+3}}\,dx$
49. $\int \dfrac{4}{4x^2+4x+65}\,dx$
50. $\int \dfrac{1}{x^2-4x+9}\,dx$
51. $\int \dfrac{1}{\sqrt{1-4x-x^2}}\,dx$
52. $\int \dfrac{12}{\sqrt{3-8x-x^2}}\,dx$

Slope Fields **In Exercises 53–56, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

53. $\dfrac{ds}{dt} = \dfrac{t}{\sqrt{1-t^4}}$, $\left(0, -\dfrac{1}{2}\right)$

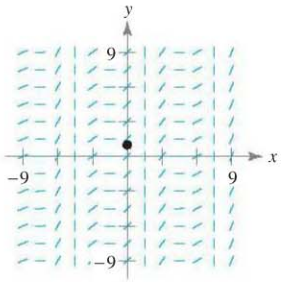


54. $\dfrac{dy}{dx} = \tan^2(2x)$, $(0, 0)$



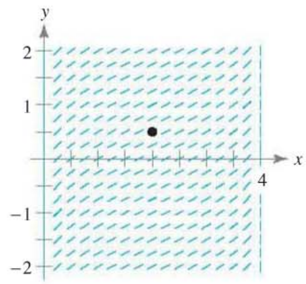
55. $\dfrac{dy}{dx} = (\sec x + \tan x)^2$

$(0, 1)$



56. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4x - x^2}}$

$\left(2, \dfrac{1}{2}\right)$



CAS ***Slope Fields*** **In Exercises 57 and 58, use a computer algebra system to graph the slope field for the differential equation and graph the solution through the specified initial condition.**

57. $\dfrac{dy}{dx} = 0.8y,\ y(0) = 4$

58. $\dfrac{dy}{dx} = 5 - y,\ y(0) = 1$

In Exercises 59–64, solve the differential equation.

59. $\dfrac{dy}{dx} = (e^x + 5)^2$

60. $\dfrac{dy}{dx} = (3 - e^x)^2$

61. $\dfrac{dr}{dt} = \dfrac{10e^t}{\sqrt{1 - e^{2t}}}$

62. $\dfrac{dr}{dt} = \dfrac{(1 + e^t)^2}{e^t}$

63. $(4 + \tan^2 x)y' = \sec^2 x$

64. $y' = \dfrac{1}{x\sqrt{4x^2 - 1}}$

In Exercises 65–72, evaluate the definite integral. Use the integration capabilities of a graphing utility to verify your result.

65. $\displaystyle\int_0^{\pi/4} \cos 2x\, dx$

66. $\displaystyle\int_0^{\pi} \sin^2 t \cos t\, dt$

67. $\displaystyle\int_0^1 xe^{-x^2}\, dx$

68. $\displaystyle\int_1^e \frac{1 - \ln x}{x}\, dx$

69. $\displaystyle\int_0^8 \frac{2x}{\sqrt{x^2 + 36}}\, dx$

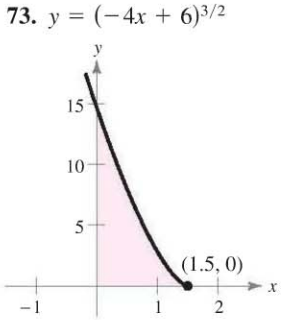
70. $\displaystyle\int_1^2 \frac{x - 2}{x}\, dx$

71. $\displaystyle\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2}\, dx$

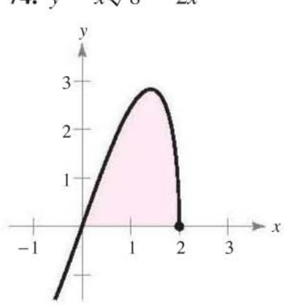
72. $\displaystyle\int_0^7 \frac{1}{\sqrt{100 - x^2}}\, dx$

Area **In Exercises 73–78, find the area of the region.**

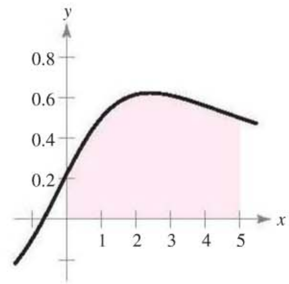
73. $y = (-4x + 6)^{3/2}$



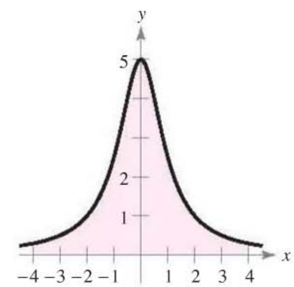
74. $y = x\sqrt{8 - 2x^2}$



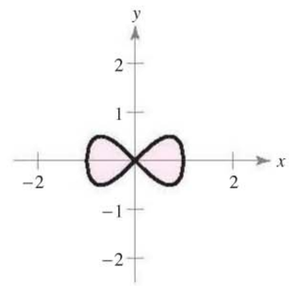
75. $y = \dfrac{3x + 2}{x^2 + 9}$



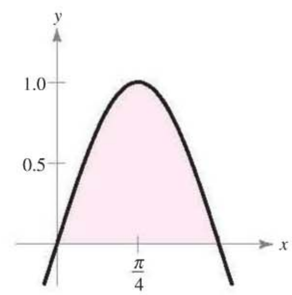
76. $y = \dfrac{5}{x^2 + 1}$



77. $y^2 = x^2(1 - x^2)$



78. $y = \sin 2x$



CAS **In Exercises 79–82, use a computer algebra system to find the integral. Use the computer algebra system to graph two antiderivatives. Describe the relationship between the graphs of the two antiderivatives.**

79. $\displaystyle\int \frac{1}{x^2 + 4x + 13}\, dx$

80. $\displaystyle\int \frac{x - 2}{x^2 + 4x + 13}\, dx$

81. $\displaystyle\int \frac{1}{1 + \sin\theta}\, d\theta$

82. $\displaystyle\int \left(\frac{e^x + e^{-x}}{2}\right)^3 dx$

WRITING ABOUT CONCEPTS

In Exercises 83–86, state the integration formula you would use to perform the integration. Explain why you chose that formula. Do not integrate.

83. $\displaystyle\int x(x^2 + 1)^3\, dx$

84. $\displaystyle\int x \sec(x^2 + 1)\tan(x^2 + 1)\, dx$

85. $\displaystyle\int \frac{x}{x^2 + 1}\, dx$

86. $\displaystyle\int \frac{1}{x^2 + 1}\, dx$

87. Determine the constants a and b such that

$\sin x + \cos x = a\sin(x + b).$

Use this result to integrate $\displaystyle\int \frac{dx}{\sin x + \cos x}$.

88. Show that $\sec x = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{1 + \sin x}$. Then use this identity to derive the basic integration rule

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C.$$

89. ***Area*** The graphs of $f(x) = x$ and $g(x) = ax^2$ intersect at the points $(0, 0)$ and $(1/a, 1/a)$. Find a $(a > 0)$ such that the area of the region bounded by the graphs of these two functions is $\frac{2}{3}$.

CAPSTONE

90. (a) Explain why the antiderivative $y_1 = e^{x+C_1}$ is equivalent to the antiderivative $y_2 = Ce^x$.

(b) Explain why the antiderivative $y_1 = \sec^2 x + C_1$ is equivalent to the antiderivative $y_2 = \tan^2 x + C$.

91. ***Think About It*** Use a graphing utility to graph the function $f(x) = \frac{1}{5}(x^3 - 7x^2 + 10x)$. Use the graph to determine whether $\int_0^5 f(x)\,dx$ is positive or negative. Explain.

92. ***Think About It*** When evaluating $\int_{-1}^{1} x^2\,dx$, is it appropriate to substitute $u = x^2$, $x = \sqrt{u}$, and $dx = \dfrac{du}{2\sqrt{u}}$ to obtain $\dfrac{1}{2}\displaystyle\int_1^1 \sqrt{u}\,du = 0$? Explain.

Approximation **In Exercises 93 and 94, determine which value best approximates the area of the region between the x-axis and the function over the given interval. (Make your selection on the basis of a sketch of the region and *not* by integrating.)**

93. $f(x) = \dfrac{4x}{x^2 + 1}$, $[0, 2]$

(a) 3 (b) 1 (c) -8 (d) 8 (e) 10

94. $f(x) = \dfrac{4}{x^2 + 1}$, $[0, 2]$

(a) 3 (b) 1 (c) -4 (d) 4 (e) 10

Interpreting Integrals **In Exercises 95 and 96, (a) sketch the region whose area is given by the integral, (b) sketch the solid whose volume is given by the integral if the disk method is used, and (c) sketch the solid whose volume is given by the integral if the shell method is used. (There is more than one correct answer for each part.)**

95. $\displaystyle\int_0^2 2\pi x^2\,dx$ **96.** $\displaystyle\int_0^4 \pi y\,dy$

97. ***Volume*** The region bounded by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = b$ $(b > 0)$ is revolved about the y-axis.

(a) Find the volume of the solid generated if $b = 1$.

(b) Find b such that the volume of the generated solid is $\frac{4}{3}$ cubic units.

98. ***Volume*** Consider the region bounded by the graphs of $x = 0$, $y = \cos x^2$, $y = \sin x^2$, and $x = \sqrt{\pi}/2$. Find the volume of the solid generated by revolving the region about the y-axis.

99. ***Arc Length*** Find the arc length of the graph of $y = \ln(\sin x)$ from $x = \pi/4$ to $x = \pi/2$.

100. ***Arc Length*** Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/3$.

101. ***Surface Area*** Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ on the interval $[0, 9]$ about the x-axis.

102. ***Centroid*** Find the x-coordinate of the centroid of the region bounded by the graphs of

$$y = \frac{5}{\sqrt{25 - x^2}}, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 4.$$

In Exercises 103 and 104, find the average value of the function over the given interval.

103. $f(x) = \dfrac{1}{1 + x^2}$, $-3 \le x \le 3$

104. $f(x) = \sin nx$, $0 \le x \le \pi/n$, n is a positive integer.

Arc Length **In Exercises 105 and 106, use the integration capabilities of a graphing utility to approximate the arc length of the curve over the given interval.**

105. $y = \tan \pi x$, $\left[0, \frac{1}{4}\right]$ **106.** $y = x^{2/3}$, $[1, 8]$

107. ***Finding a Pattern***

(a) Find $\displaystyle\int \cos^3 x\,dx$. (b) Find $\displaystyle\int \cos^5 x\,dx$.

(c) Find $\displaystyle\int \cos^7 x\,dx$.

(d) Explain how to find $\int \cos^{15} x\,dx$ without actually integrating.

108. ***Finding a Pattern***

(a) Write $\int \tan^3 x\,dx$ in terms of $\int \tan x\,dx$. Then find $\int \tan^3 x\,dx$.

(b) Write $\int \tan^5 x\,dx$ in terms of $\int \tan^3 x\,dx$.

(c) Write $\int \tan^{2k+1} x\,dx$, where k is a positive integer, in terms of $\int \tan^{2k-1} x\,dx$.

(d) Explain how to find $\int \tan^{15} x\,dx$ without actually integrating.

109. ***Methods of Integration*** Show that the following results are equivalent.

Integration by tables:

$$\int \sqrt{x^2 + 1}\,dx = \frac{1}{2}\left(x\sqrt{x^2 + 1} + \ln\left|x + \sqrt{x^2 + 1}\right|\right) + C$$

Integration by computer algebra system:

$$\int \sqrt{x^2 + 1}\,dx = \frac{1}{2}\left[x\sqrt{x^2 + 1} + \operatorname{arcsinh}(x)\right] + C$$

PUTNAM EXAM CHALLENGE

110. Evaluate $\displaystyle\int_2^4 \frac{\sqrt{\ln(9 - x)}\,dx}{\sqrt{\ln(9 - x)} + \sqrt{\ln(x + 3)}}$.

This problem was composed by the Committee on the Putnam Prize Competition.